

The generation problem in Thompson group F

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Abstract

We show that the generation problem in Thompson group F is decidable, i.e., there is an algorithm which decides if a finite set of elements of F generates the whole F . The algorithm makes use of the Stallings 2-core of subgroups of F , which can be defined in an analogue way to the Stallings core of subgroups of a finitely generated free group. Further study of the Stallings 2-core of subgroups of F provides a solution to another algorithmic problem in F . Namely, given a finitely generated subgroup H of F , it is decidable if H acts transitively on the set of finite dyadic fractions \mathcal{D} . Other applications of the study include the construction of new maximal subgroups of F of infinite index, among which, a maximal subgroup of infinite index which acts transitively on the set \mathcal{D} and the construction of an elementary amenable subgroup of F which is maximal in a normal subgroup of F .

Contents

1	Introduction	2
2	Preliminaries on F	6
2.1	F as a group of homeomorphisms	6
2.2	F as a diagram group	6
2.2.A	Directed complexes and diagram groups	6
2.2.B	A normal form of elements of F	10
2.3	The relation between F as a diagram group and F as a group of homeomorphisms	11
2.4	On orbitals and stabilizers in F	13
2.5	The derived subgroup of F	14
3	The Stallings 2-core of subgroups of diagram groups	14
4	Paths on the core of a subgroup $H \leq F$	19
5	The closure of a subgroup $H \leq F$	24

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6	Transitivity of the action of H on the set \mathcal{D}	29
7	The generation problem in F	34
8	The Tuples algorithm	43
9	F is a cyclic extension of a subgroup K which has a maximal elementary amenable subgroup	49
10	Computations related to $\mathcal{L}(H)$	51
10.1	On the algorithm for finding a generating set of $\text{Cl}(H)$	51
10.2	Core 2-automata	57
10.3	Maximal subgroups of F of infinite index	63
10.3.A	Construction of maximal subgroups of F of infinite index which do not fix any number in $(0, 1)$	63
10.3.B	A maximal subgroup of F which acts transitively on the set \mathcal{D}	67
11	Solvable subgroups of Thompson group F	68
11.1	On the closure of solvable subgroups	68
11.2	Characterization of solvable subgroups $H \leq F$ in terms of the core $\mathcal{L}(H)$. . .	73
12	Open problems	78
12.1	Subgroups of F whose closure contains the derived subgroup of F	78
12.2	Maximal subgroups of F of infinite index	78
12.2.A	Closed maximal subgroups	78
12.2.B	The action of maximal subgroups of infinite index in F on $[0, 1]$	78
12.2.C	2-generated maximal subgroups of F	79
12.3	Subgroups $H \leq F$ with finite core $\mathcal{L}(H)$	80

1 Introduction

Recall that R. Thompson's group F is the group of all piecewise-linear homeomorphisms of the interval $[0, 1]$ with finitely many breakpoints, where all breakpoints are finite dyadic and all slopes are integer powers of 2. The group F has a presentation with two generators and two defining relations [11, 25] (see below).

Decision problems in F have been extensively studied. It is well known that the word problem in F is decidable in linear time [11, 28, 17] and that the conjugacy problem is decidable in linear time [17, 1, 2]. The simultaneous conjugacy problem [21] and twisted conjugacy problem [10] have also been proven to be decidable. In [7], an algorithm for deciding if a finitely generated subgroup H of F is solvable, is given. On the other hand, it is proved in [10] that there are orbit undecidable subgroups of $\text{Aut}(F)$ and hence, there are extensions of Thompson's group F by finitely generated free groups, with unsolvable conjugacy problem.

In this paper we consider the generation problem in Thompson group F . Namely, the problem of deciding for a given finite subset X of F whether it generates the whole F . Note

that the solvability of the membership problem for subgroups in F is a very interesting open problem (it is mentioned in [18]) and the generation problem is an important special case of the membership problem (we need to check if the generators of F are in the subgroup). The generation problem is known to be undecidable for $F_2 \times F_2$ (see, for example, [22]). Using Rips' construction [24], one can find a hyperbolic group G which projects onto $F_2 \times F_2$ and has undecidable generation problem. Moreover, using Wise's version of Rips' construction [30], one can ensure that G is linear over \mathbb{Z} .

We prove that the generation problem in Thompson group F is decidable. The algorithm solving the generation problem makes use of the definition of F as a diagram group [17] and the construction of the Stallings 2-core of subgroups of diagram groups. The construction is due to Guba and Sapir from 1999, but appeared in print first in [15].

Every element of F can be viewed as a diagram Δ . A diagram Δ in F is a directed labeled plane graph tessellated by cells, defined up to an isotopy of the plane. It has one top edge and one bottom edge and the whole diagram is situated between them. Every cell in Δ is either a positive or a negative cell. Positive cells have one top edge and two bottom edges. Negative cells are the reflection of positive cells about a horizontal line (See Figure 2.3). In particular, a diagram Δ can be naturally viewed as a directed 2-complex; i.e., a 2-complex which consists of vertices, directed edges and 2-cells bounded by two directed paths with the same endpoints.

The Stallings 2-core $\mathcal{L}(H)$ of a subgroup $H \leq F$ can be viewed as a 2-dimensional analogue of the Stallings core of subgroups of free groups (see [29]). The Stallings 2-core $\mathcal{L}(H)$ is a directed 2-complex associated with H which has a distinguished input/output edge. We say that $\mathcal{L}(H)$ is a *2-automaton*. The core $\mathcal{L}(H)$ *accepts* a diagram Δ in F if there is a morphism of directed 2-complexes from Δ to $\mathcal{L}(H)$ which maps the top and bottom edges of Δ to the input/output edge of $\mathcal{L}(H)$.

By construction [15], $\mathcal{L}(H)$ accepts all diagrams in H , but unlike in the case of free groups, the core $\mathcal{L}(H)$ can accept diagrams not in H . We define the *closure* of H to be the subgroup of F of all diagrams accepted by the core $\mathcal{L}(H)$. The closure operation satisfies the usual properties of closure. Namely, $H \leq \text{Cl}(H)$, $\text{Cl}(\text{Cl}(H)) = \text{Cl}(H)$ and if $H_1 \leq H_2$ then $\text{Cl}(H_1) \leq \text{Cl}(H_2)$. We say that H is *closed* if $H = \text{Cl}(H)$. The closure of H is a diagram group over the directed 2-complex $\mathcal{L}(H)$. Thus, if H is finitely generated, the membership problem in $\text{Cl}(H)$ is decidable [17].

Given a finitely generated subgroup H of F , one can check if the generators of F are accepted by $\mathcal{L}(H)$. If not, then $\text{Cl}(H)$, and in particular H , is a proper subgroup of F . This however, only gives a partial solution to the generation problem in F . Indeed, there are finitely generated proper subgroups H of F such that $\text{Cl}(H) = F$.

To solve the generation problem in F , we study the core and the closure of subgroups H of F . The first result in the study is the following characterization of the closure of subgroups of F .

Theorem 1.1. Let $H \leq F$. Then $\text{Cl}(H)$ is the subgroup of F consisting of piecewise-linear functions f , with finitely many pieces, such that on each piece, f coincides with the restriction of some function from H .

Thus $\text{Cl}(H)$ can be defined “dynamically” as the topological full group (see, for example, [13]) of the group H acting on the set of finite dyadic fractions of the unit interval $(0, 1)$ with

the natural topology.

Theorem 1.1, proves a conjecture of Guba and Sapir about the closure of subgroups of diagram groups (see Conjecture 3.11 below), in the special case of Thompson group F . It follows from Theorem 1.1 that the orbits of the action of H on the interval $[0, 1]$ coincide with the orbits of the action of $\text{Cl}(H)$.

The proof of Theorem 1.1 enables us to get the following characterization of subgroups $H \leq F$ which act transitively on the set of finite dyadic fractions \mathcal{D} . Cells in $\mathcal{L}(H)$, as in diagrams Δ , are either positive or negative. Here as well, positive cells have one top edge and two bottom edges. The core $\mathcal{L}(H)$ has natural initial and terminal vertices. Every other vertex is an *inner vertex* of $\mathcal{L}(H)$.

Theorem 1.2. Let $H \leq F$. Then H acts transitively on the set of finite dyadic fractions \mathcal{D} if and only if the following conditions are satisfied.

- (1) Every edge in $\mathcal{L}(H)$ is the top edge of some positive cell.
- (2) There is a unique inner vertex in $\mathcal{L}(H)$.

An *inner edge* is an edge of $\mathcal{L}(H)$ whose endpoints are inner vertices of $\mathcal{L}(H)$. We observe that if one replaces “inner vertex” in Theorem 1.2 by “inner edge”, one gets the criterion for $\text{Cl}(H)$ to contain $[F, F]$ (Lemma 7.1 below). To solve the generation problem in F we give a criterion for the group H to contain the derived subgroup of F (equivalently, for H to be a normal subgroup of F [11]). It turns out that one only has to add a somewhat technical condition to the requirement that $\text{Cl}(H)$ contains $[F, F]$.

Theorem 1.3. Let $H \leq F$. Then H contains the derived subgroup of F if and only if the following conditions hold.

- (1) $[F, F] \subseteq \text{Cl}(H)$
- (2) There is a function $h \in H$ which fixes a finite dyadic fraction $\alpha \in (0, 1)$ such that the slope $h'(\alpha^-) = 1$ and the slope $h'(\alpha^+) = 2$.

Given a finite subset X of F , we let H be the group generated by X . Then it is (easily) decidable if condition (1) holds for H . In Section 8, we give an algorithm for deciding if H satisfies condition (2), given that H satisfies condition (1). As one can also decide if $H[F, F] = F$, Theorem 1.3 gives a solution for the generation problem in Thompson group F .

Corollary 1.4. Let H be a subgroup of F . Then $H = F$ if and only if the following conditions hold.

- (1) $[F, F] \subseteq \text{Cl}(H)$
- (2) $H[F, F] = F$.
- (3) There is a function $h \in H$ which fixes a finite dyadic fraction $\alpha \in (0, 1)$ such that the slope $h'(\alpha^-) = 1$ and the slope $h'(\alpha^+) = 2$.

Another application of Theorem 1.3 is the following.

Theorem 1.5. There is a sequence of finitely generated subgroups $B < K < F$ such that B is elementary amenable and maximal in K , K is normal in F and F/K is infinite cyclic.

In Section 10 we give some techniques related to the core $\mathcal{L}(H)$ of a subgroup H of F . In Section 10.1 we consider the problem of finding a generating set of $\text{Cl}(H)$, and show that an algorithm due to Guba and Sapir [17] for finding a generating set of a diagram group, can be simplified in that special case. In Section 10.2 we give conditions for a 2-automaton \mathcal{L} over the Duncie hat \mathcal{K} (see Section 2.2.A) to coincide with the core $\mathcal{L}(H)$ of some subgroup H of F , up to identification of vertices. These techniques are applied in Section 10.3 to the construction of maximal subgroups of infinite index in F .

In [26, 27] Dmytro Savchuk studied subgroups H_U of the group F which are the stabilizers of finite sets of real numbers $U \subset (0, 1)$. He proved that if U consists of one number, then H_U is a maximal subgroup of F . He also showed that the Schreier graphs of the subgroups H_U are amenable. He asked [27, Problem 1.5] whether every maximal subgroup of infinite index in F is of the form $H_{\{\alpha\}}$, that is, fixes a number from $(0, 1)$. In [15], the author and Sapir applied the core of subgroups of F to prove the existence of maximal subgroups of F of infinite index which do not fix any number in $(0, 1)$. An explicit example of a 3-generated maximal subgroup that does not fix any number in $(0, 1)$ was also constructed in [15], but all the examples from [15] stabilize proper subsets of \mathcal{D} . Applying Corollary 1.4, Theorem 1.2 and the techniques from Section 10 we prove the following.

Theorem 1.6. Thompson group F has a 3-generated maximal subgroup of infinite index which acts transitively on the set of finite dyadic fractions in $(0, 1)$.

In Section 11.1, we consider the closure of solvable subgroups H of F and prove the following theorem.

Theorem 1.7. Let H be a solvable subgroup of F of derived length n . Then the following assertions hold.

- (1) The action of H on the set of finite dyadic fractions \mathcal{D} has infinitely many orbits.
- (2) $\text{Cl}(H)$ is solvable of derived length n .
- (3) If H is finitely generated then $\text{Cl}(H)$ is finitely generated.

The theorem follows from results about solvable subgroups of Thompson group F (and more generally, solvable subgroups of $\text{PL}_o(I)$, the group of piecewise linear orientation preserving homeomorphisms of the unit interval $[0, 1]$ with finitely many pieces) from [4, 5] and [7].

In Section 11.2, we give a characterization of solvable subgroups H of F in terms of the core $\mathcal{L}(H)$. We define a directed graph $\mathcal{P}(H)$ related to the core $\mathcal{L}(H)$ (Definition 11.21) and prove the following.

Theorem 1.8. Let H be a subgroup of F . Then H is solvable if and only if there is a uniform upper bound on the lengths of all directed paths in $\mathcal{P}(H)$. If H is solvable then the derived length of H is the maximal length of a directed path in $\mathcal{P}(H)$.

Theorem 1.8 makes use of the characterization of solvable subgroups of F in terms of their towers (see Definition 11.4 below) due to Bleak [4]. When H is finitely generated, Theorem 1.8 translates to a simple algorithm for deciding if H is solvable. As mentioned above, there is an algorithm for determining the solvability of finitely generated subgroups of F due to Bleak, Brough and Hermiller [7]. In fact, the algorithm from [7] applies to all computable

finitely generated subgroups of $\text{PL}_o(I)$ (see [7, Section 4]). For finitely generated subgroups of F , the algorithm given by Theorem 1.8 is arguably easier than the one in [7].

In Section 12, we list some open problems.

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2 Preliminaries on F

2.1 F as a group of homeomorphisms

Recall that F consists of all piecewise-linear increasing self-homeomorphisms of the unit interval with slopes of all linear pieces powers of 2 and all break points of the derivative finite dyadic fractions. The group F is generated by two functions x_0 and x_1 defined as follows [11].

$$x_0(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad x_1(t) = \begin{cases} t & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2t - \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq \frac{5}{8} \\ t + \frac{1}{8} & \text{if } \frac{5}{8} \leq t \leq \frac{3}{4} \\ \frac{t}{2} + \frac{1}{2} & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$$

The composition in F is from left to right.

Every element of F is completely determined by how it acts on the set of finite dyadic fractions. Every number in $(0, 1)$ can be described as $.s$ where s is an infinite word in $\{0, 1\}$. For each element $g \in F$ there exists a finite collection of pairs of (finite) words (u_i, v_i) in the alphabet $\{0, 1\}$ such that every infinite word in $\{0, 1\}$ starts with exactly one of the u_i 's. The action of F on a number $.s$ is the following: if s starts with u_i , we replace u_i by v_i . For example, x_0 and x_1 are the following functions:

$$x_0(t) = \begin{cases} .0\alpha & \text{if } t = .00\alpha \\ .10\alpha & \text{if } t = .01\alpha \\ .11\alpha & \text{if } t = .1\alpha \end{cases} \quad x_1(t) = \begin{cases} .0\alpha & \text{if } t = .0\alpha \\ .10\alpha & \text{if } t = .100\alpha \\ .110\alpha & \text{if } t = .101\alpha \\ .111\alpha & \text{if } t = .11\alpha \end{cases}$$

For the generators x_0, x_1 defined above, the group F has the following finite presentation [11].

$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_1^{x_0}] = 1, [x_0 x_1^{-1}, x_1^{x_0^2}] = 1 \rangle,$$

where a^b denotes $b^{-1}ab$.

2.2 F as a diagram group

2.2.A Directed complexes and diagram groups

The definition of F we will use most often in this paper is that of F as a diagram group. This section closely follows [19].

Definition 2.1. For every directed graph Γ let \mathcal{P} be the set of all (directed) paths in Γ , including the empty paths. A *directed 2-complex* is a directed graph Γ equipped with a set \mathcal{F} (called the *set of 2-cells*), and three maps $\mathbf{top} \cdot : \mathcal{F} \rightarrow \mathcal{P}$, $\mathbf{bot} \cdot : \mathcal{F} \rightarrow \mathcal{P}$, and $^{-1} : \mathcal{F} \rightarrow \mathcal{F}$ called *top*, *bottom*, and *inverse* such that

- for every $f \in \mathcal{F}$, the paths $\mathbf{top}(f)$ and $\mathbf{bot}(f)$ are non-empty and have common initial vertices and common terminal vertices,
- $^{-1}$ is an involution without fixed points, and $\mathbf{top}(f^{-1}) = \mathbf{bot}(f)$, $\mathbf{bot}(f^{-1}) = \mathbf{top}(f)$ for every $f \in \mathcal{F}$.

We will usually assume that \mathcal{F} is given with an orientation, that is, a subset $\mathcal{F}^+ \subseteq \mathcal{F}$ of *positive* 2-cells, such that \mathcal{F} is the disjoint union of \mathcal{F}^+ and the set $\mathcal{F}^- = (\mathcal{F}^+)^{-1}$ (the latter is called the set of *negative* 2-cells).

If \mathcal{K} is a directed 2-complex, then paths on \mathcal{K} are called *1-paths* (we are going to have 2-paths later). The initial and terminal vertex of a 1-path p are denoted by $\iota(p)$ and $\tau(p)$, respectively. For every 2-cell $f \in \mathcal{F}$, the vertices $\iota(\mathbf{top}(f)) = \iota(\mathbf{bot}(f))$ and $\tau(\mathbf{top}(f)) = \tau(\mathbf{bot}(f))$ are denoted $\iota(f)$ and $\tau(f)$, respectively.

We shall denote each cell f by $\mathbf{top}(f) \rightarrow \mathbf{bot}(f)$. And we can denote a directed 2-complex \mathcal{K} similar to a semigroup presentation $\langle E \mid \mathbf{top}(f) \rightarrow \mathbf{bot}(f), f \in \mathcal{F}^+ \rangle$ where E is the set of all edges of \mathcal{K} (note that we ignore the vertices of \mathcal{K}).

For example, the directed 2-complex $\langle x \mid x \rightarrow x^2 \rangle$ is the *Dunce hat* obtained by identifying all edges in the triangle (Figure 2.1) according to their directions. It has one vertex, one edge, and one positive 2-cell.

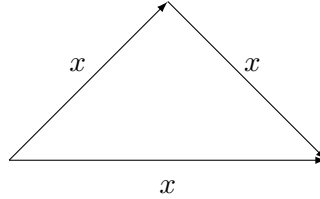


Figure 2.1: Dunce hat

With the directed 2-complex \mathcal{K} , one can associate a category $\Pi(\mathcal{K})$ whose objects are directed 1-paths, and morphisms are *2-paths*, i.e. sequences of replacements of $\mathbf{top}(f)$ by $\mathbf{bot}(f)$ in 1-paths, $f \in \mathcal{F}$. More precisely, an *atomic 2-path* (an *elementary homotopy*) is a triple (p, f, q) , where p, q are 1-paths in \mathcal{K} , and $f \in \mathcal{F}$ such that $\tau(p) = \iota(f)$, $\tau(f) = \iota(q)$. If δ is the atomic 2-path (p, f, q) , then $p\mathbf{top}(f)q$ is denoted by $\mathbf{top}(\delta)$, and $p\mathbf{bot}(f)q$ is denoted by $\mathbf{bot}(\delta)$; these are called the *top* and the *bottom* 1-paths of the atomic 2-path. Every *nontrivial* 2-path δ on \mathcal{K} is a sequence of atomic paths $\delta_1, \dots, \delta_n$, where $\mathbf{bot}(\delta_i) = \mathbf{top}(\delta_{i+1})$ for every $1 \leq i < n$. In this case n is called the *length* of the 2-path δ . The *top* and the *bottom* 1-paths of δ , denoted by $\mathbf{top}(\delta)$ and $\mathbf{bot}(\delta)$, are $\mathbf{top}(\delta_1)$ and $\mathbf{bot}(\delta_n)$, respectively. Every 1-path p is considered as a trivial 2-path with $\mathbf{top}(p) = \mathbf{bot}(p) = p$. These are the identity morphisms in the category $\Pi(\mathcal{K})$. The composition of 2-paths δ and δ' is called *concatenation* and is denoted $\delta \circ \delta'$.

With every atomic 2-path $\delta = (p, f, q)$, where $\mathbf{top}(f) = u$, $\mathbf{bot}(f) = v$ we associate the labeled plane graph Δ in Figure 2.2. An arc labeled by a word w is subdivided into $|w|$ edges. All edges are oriented from left to right. The label of each oriented edge of the graph is a symbol from the alphabet E , the set of edges in \mathcal{K} . As a plane graph, it has only one bounded face; we label it by the corresponding cell f of \mathcal{K} . This plane graph Δ is called the diagram of δ . Such diagrams are called *atomic*. The leftmost and rightmost vertices of Δ are denoted by $\iota(\Delta)$ and $\tau(\Delta)$, respectively. The diagram Δ has two distinguished paths from $\iota(\Delta)$ to $\tau(\Delta)$ that correspond to the top and bottom paths of Δ . Their labels are puq and pvq , respectively. These are called the top and the bottom paths of Δ denoted by $\mathbf{top}(\Delta)$ and $\mathbf{bot}(\Delta)$.

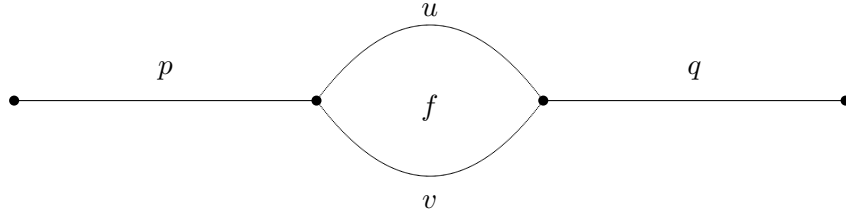


Figure 2.2: An atomic diagram

The diagram corresponding to the trivial 2-path p is just an arc labeled by p ; it is called a *trivial diagram* and it is denoted by $\varepsilon(p)$.

Let $\delta = \delta_1 \circ \delta_2 \circ \dots \circ \delta_n$ be a 2-path in \mathcal{K} , where $\delta_1, \dots, \delta_n$ are atomic 2-paths. Let Δ_i be the atomic diagram corresponding to δ_i . Then the bottom path of Δ_i has the same label as the top path of Δ_{i+1} ($1 \leq i < n$). Hence we can identify the bottom path of Δ_i with the top path of Δ_{i+1} for all $1 \leq i < n$, and obtain a plane graph Δ , which is called the *diagram of the 2-path* δ . If the top path of Δ is labeled u and the bottom path is labeled v , then we say that Δ is a (u, v) -diagram. Similarly, a cell π in a diagram Δ is a (u, v) -cell if it is labeled by a 2-cell f of \mathcal{K} with $\mathbf{top}(f) = u$ and $\mathbf{bot}(f) = v$. If Δ is a diagram of some 2-path δ in \mathcal{K} , we say that Δ is a diagram over \mathcal{K} .

Two diagrams Δ_1, Δ_2 are considered *equal* if they are isotopic as plane graphs. In that case, we write $\Delta_1 \equiv \Delta_2$. The isotopy must take vertices to vertices, edges to edges, it must also preserve labels of edges and cells. Two 2-paths are called *isotopic* if the corresponding diagrams are equal.

Concatenation of 2-paths corresponds to the concatenation of diagrams: if the bottom path of Δ_1 and the top path of Δ_2 have the same labels, we can identify them and obtain a new diagram $\Delta_1 \circ \Delta_2$.

Note that for any atomic 2-path $\delta = (p, f, q)$ in \mathcal{K} one can naturally define its *inverse* 2-path $\delta^{-1} = (p, f^{-1}, q)$. The inverses of all 2-paths and diagrams are defined naturally. The inverse diagram Δ^{-1} of Δ is obtained by taking the mirror image of Δ with respect to a horizontal line, and replacing labels of cells by their inverses.

Let us identify in the category $\Pi(\mathcal{K})$ all isotopic 2-paths and also identify each 2-path of the form $\delta' \delta \delta^{-1} \delta''$ with $\delta' \delta''$. The quotient category is obviously a groupoid (i.e. a category with invertible morphisms). It is denoted by $\mathcal{D}(\mathcal{K})$ and is called the *diagram groupoid* of

\mathcal{K} . Two 2-paths are called *homotopic* if they correspond to the same morphism in $\mathcal{D}(\mathcal{K})$. For each 1-path p of \mathcal{K} , the local group of $\mathcal{D}(\mathcal{K})$ at p (i.e., the group of homotopy classes of 2-paths connecting p with itself) is called the *diagram group of the directed 2-complex \mathcal{K} with base p* and is denoted by $\text{DG}(\mathcal{K}, p)$.

The following theorem is proved in [17] (see also [19]).

Theorem 2.2. If \mathcal{K} is the Duncce hat (see Figure 2.1) and x is the edge of it, then $\text{DG}(\mathcal{K}, x)$ is isomorphic to the R. Thompson group F . The generators x_0, x_1 of F are depicted in Figure 2.3 (all edges in the diagrams are labeled by x and oriented from left to right).

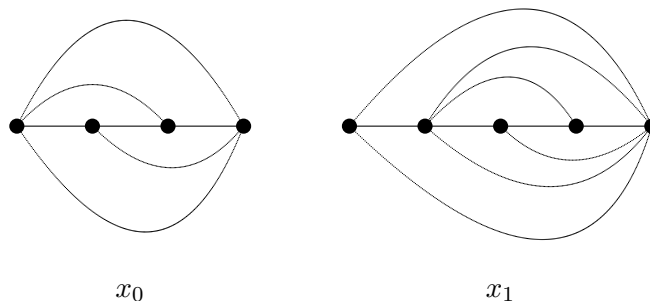


Figure 2.3: Generators of the R. Thompson group F

Diagrams Δ_1, Δ_2 over \mathcal{K} corresponding to homotopic 2-paths are called *equivalent* (denoted by $\Delta_1 = \Delta_2$). The equivalence relation on the set of diagrams (and the homotopy relation on the set of 2-paths of \mathcal{K}) can be defined very easily as follows. We say that two cells π_1 and π_2 in a diagram Δ over \mathcal{K} form a *dipole* if $\text{bot}(\pi_1)$ coincides with $\text{top}(\pi_2)$ and the labels of the cells π_1 and π_2 are mutually inverse. Clearly, if π_1 and π_2 form a dipole, then one can remove the two cells from the diagram and identify $\text{top}(\pi_1)$ with $\text{bot}(\pi_2)$. The result will be some diagram Δ' . As in [17], it is easy to prove that if δ is a 2-path corresponding to Δ , then the diagram Δ' corresponds to a 2-path δ' , which is homotopic to δ . We call a diagram *reduced* if it does not contain dipoles. A 2-path δ in \mathcal{K} is called *reduced* if the corresponding diagram is reduced.

Thus one can define morphisms in the diagram groupoid $\mathcal{D}(\mathcal{K})$ as reduced diagrams over \mathcal{K} with operation “concatenation + reduction” (that is, the product of two reduced diagrams Δ and Δ' is the result of removing all dipoles from $\Delta \circ \Delta'$ step by step; that process is confluent and terminating, so the result is uniquely determined [17, Lemma 3.10]). Thus, for each 1-path u , the diagram group $\text{DG}(\mathcal{K}, u)$, is composed of all reduced (u, u) -diagrams over \mathcal{K} . We would often consider a non-reduced diagram as an element of $\text{DG}(\mathcal{K}, u)$ identified with the reduced diagram equivalent to it.

One can naturally define an addition operation in the diagram groupoid $\mathcal{D}(\mathcal{K})$. Let Δ_1 and Δ_2 be diagrams over \mathcal{K} with $\text{top}(\Delta_1)$ labeled u and $\text{top}(\Delta_2)$ labeled v . If the 1-paths u and v in \mathcal{K} satisfy $\tau(u) = \iota(v)$, then one can identify $\tau(\Delta_1)$ and $\iota(\Delta_2)$ to get a new diagram over \mathcal{K} , $\Delta_1 + \Delta_2$. Note that if \mathcal{K} has only one vertex, then the addition operation in $\mathcal{D}(\mathcal{K})$ is everywhere defined.

Remark 2.3 ([19]). Let \mathcal{K} be a directed 2-complex and let \mathcal{K}' be a directed 2-complex resulting from \mathcal{K} by identification of vertices. Let u be a 1-path in \mathcal{K} . In particular, u is also a 1-path in \mathcal{K}' . Then the diagram groups $\text{DG}(\mathcal{K}, u)$ and $\text{DG}(\mathcal{K}', u)$ coincide.

Remark 2.3 is the reason we usually ignore vertices in the description of a directed 2-complex \mathcal{K} . We will follow this tradition in this paper. On occasion however, the vertices of \mathcal{K} will be important to us. On those occasions we will be careful to distinguish between different vertices.

2.2.B A normal form of elements of F

Let x_0, x_1 be the standard generators of F . Recall that $x_{i+1}, i \geq 1$, denotes $x_0^{-i} x_1 x_0^i$. In these generators, the group F has the following presentation $\langle x_i, i \geq 0 \mid x_i^{x_j} = x_{i+1} \text{ for every } j < i \rangle$ [11].

There exists a clear connection between representation of elements of F by diagrams and the normal form of elements in F . Recall [11] that every element in F is uniquely representable in the following form:

$$x_{i_1}^{s_1} \dots x_{i_m}^{s_m} x_{j_n}^{-t_n} \dots x_{j_1}^{-t_1}, \quad (2.1)$$

where $i_1 \leq \dots \leq i_m \neq j_n \geq \dots \geq j_1$; $s_1, \dots, s_m, t_1, \dots, t_n \geq 1$, and if x_i and x_i^{-1} occur in (2.1) for some $i \geq 0$ then either x_{i+1} or x_{i+1}^{-1} also occurs in (2.1). This form is called the *normal form* of elements in F .

Let \mathcal{K} be the Dunce hat and let x be the edge of \mathcal{K} . Every cell in a diagram Δ over \mathcal{K} is either an (x, x^2) -cell or an (x^2, x) -cell. A dipole $\pi_1 \circ \pi_2$ in Δ is a dipole of *type 1* if π_1 is an (x, x^2) -cell and π_2 is an (x^2, x) -cell. Otherwise, the dipole is of *type 2*. Directed paths in Δ will be called 1-paths. It was noticed in [15] that if Δ is a diagram over \mathcal{K} with no dipoles of type 2, then Δ is divided by its *horizontal 1-path*; i.e., the longest 1-path from $\iota(\Delta)$ to $\tau(\Delta)$, into two parts, *positive* and *negative*, denoted by Δ^+ and Δ^- respectively. So $\Delta \equiv \Delta^+ \circ \Delta^-$. It is easy to prove by induction on the number of cells that all cells in Δ^+ are (x, x^2) -cells and all cells in Δ^- are (x^2, x) -cells.

Let us show how given a reduced diagram Δ in $\text{DG}(\mathcal{K}, x)$ one can get the normal form of the element of F represented by this diagram. This is the left-right dual of the procedure described in [15, Example 2] and after Theorem 5.6.41 in [25].

Lemma 2.1. *Let us number the cells of Δ^+ by numbers from 1 to k by taking every time the “leftmost” cell, that is, the cell which is to the left of any other cell attached to the bottom path of the diagram formed by the previous cells. The first cell is attached to the top path of Δ^+ (which is the top path of Δ). The i^{th} cell in this sequence of cells corresponds to an atomic diagram, which has the form $(x^{\ell_i}, x \rightarrow x^2, x^{r_i})$, where ℓ_i (r_i) is the length of the path from the initial (resp. terminal) vertex of the diagram (resp. the cell) to the initial (resp. terminal) vertex of the cell (resp. the diagram), such that the path is contained in the bottom path of the diagram formed by the first $i - 1$ cells. If $r_i = 0$ then we label this cell by 1. If $r_i \neq 0$ then we label this cell by the element x_{ℓ_i} of F . Multiplying the labels of all cells, we get the “positive” part of the normal form. In order to find the “negative” part of the normal form, consider $(\Delta^-)^{-1}$, number its cells as above and label them as above. The normal form*

of Δ is then the product of the normal form of Δ^+ and the inverse of the normal form of $(\Delta^-)^{-1}$.

For example, applying the procedure from Lemma 2.1 to the diagram on Figure 2.4 we

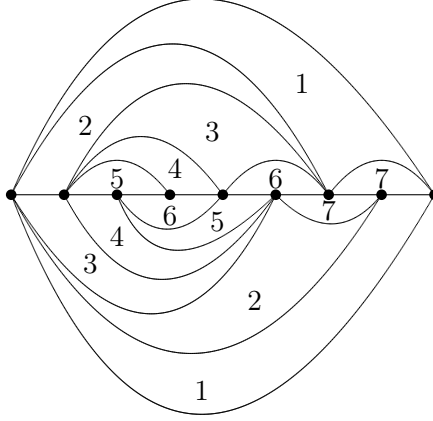


Figure 2.4: Reading the normal form of an element of F off its diagram.

get the normal form $x_0 x_1^3 x_4 (x_6^2 x_1 x_2^2 x_5)^{-1}$.

2.3 The relation between F as a diagram group and F as a group of homeomorphisms

One can define an isomorphism from F viewed as a diagram group to F as a group of homeomorphisms of $[0, 1]$. To do so, we define pairs of branches of a diagram Δ .

Let Δ be a diagram in F , i.e., a diagram in $\text{DG}(\mathcal{K}, x)$, where \mathcal{K} is the Duncie hat. All diagrams in F considered in this paper will be assumed to have no dipoles of type 2. Thus, we will not mention it when considering a diagram Δ in F . Since Δ has no dipoles of type 2, $\Delta \equiv \Delta^+ \circ \Delta^-$.

A *tree-diagram* is an (x, x^n) -diagram Ψ over \mathcal{K} for some $n \in \mathbb{N}$ where all cells are positive cells. We note that if Δ is a diagram in F with no dipoles of type 2 then Δ^+ is a tree-diagram and so is $(\Delta^-)^{-1}$.

Given a tree-diagram Ψ , one can put a vertex at the middle of each edge of Ψ and inside every cell π of Ψ draw two edges; from the vertex on **top**(π) to the vertex on the left bottom edge of π and from the vertex on **top**(π) to the vertex on the right bottom edge of π . The result is a finite binary tree with n leaves, where n is the number of edges in **bot**(Ψ). A *path* on a rooted binary tree is a directed simple path starting from the root. A *branch* in a binary tree is a maximal path. That gives rise to the following definition of paths on tree-diagrams. Note the difference from 1-paths on diagrams defined above.

Definition 2.4. Let Ψ be a tree-diagram over \mathcal{K} . A *path* on Ψ is a sequence of edges e_1, \dots, e_m such that

- (1) $e_1 = \mathbf{top}(\Psi)$.
- (2) For each $i = 1, \dots, m-1$, there is an (x, x^2) -cell π_i in Ψ with $\mathbf{top}(\pi_i) = e_i$ and such that e_{i+1} is either the left or the right bottom edge of π_i .

A maximal path on Ψ is called a *branch* of Ψ . If $p = e_1, \dots, e_m$ is a path on Ψ , then the *label* $\text{lab}(p)$ of p is defined to be a binary word u of length $m-1$, $u \equiv u_1 \cdots u_{m-1}$ ¹, where for each i , the letter $u_i \equiv 0$ if it corresponds to a step to the left (i.e., if e_{i+1} is the left bottom edge of the cell π_i); $u_i \equiv 1$ if it corresponds to a step to the right.

If Ψ is a tree-diagram, paths on Ψ^{-1} are defined in a similar way. The initial edge of a path on Ψ^{-1} is $\mathbf{bot}(\Psi^{-1})$. If Δ is a diagram in F (with no dipoles of type 2) then paths on Δ^+ are said to be *positive paths* on Δ . Paths on Δ^- are *negative paths* on Δ . Positive and negative branches of Δ are defined in a similar way.

A path p on a tree-diagram Ψ is uniquely determined by its label u . Thus, we will often consider the path and its label as the same object. If p is a path on Ψ and $\text{lab}(p) \equiv u$, we will denote by p^+ or by u^+ the last edge in the path. Since for each branch p of Ψ the terminal edge p^+ lies on the path $\mathbf{bot}(\Psi)$, the branches of Ψ are naturally ordered from left to right. If p is a positive (resp. negative) path in a diagram Δ , then the terminal edge p^+ is an edge of Δ^+ (resp. Δ^-). If p is a positive or negative branch of Δ , then p^+ lies on the horizontal 1-path of Δ .

Let Ψ be a tree-diagram over \mathcal{K} . We make the following observation about consecutive branches in Ψ . It will be used often throughout the paper with no specific reference.

Remark 2.5. Let u_1 and u_2 be (the labels of) consecutive branches of Ψ . Let u be the longest common prefix of u_1 and u_2 (u can be empty). Then

$$u_1 \equiv u01^m \quad \text{and} \quad u_2 \equiv u10^n$$

for some $m, n \geq 0$.

Let Δ be a diagram in F . Let u_1, \dots, u_n be the (labels of the) positive branches of Δ and v_1, \dots, v_n be the (labels of the) negative branches of Δ , ordered from left to right. For each $i = 1, \dots, n$, we say that Δ has a *pair of branches* $u_i \rightarrow v_i$. The function g from F corresponding to this diagram takes binary fraction $.u_i\omega$ to $.v_i\omega$ for every i and every infinite binary word ω . We will also say that the element g takes the branch u_i to the branch v_i .

If e is an edge on the horizontal 1-path of Δ , then one can replace e by a dipole $\pi_1 \circ \pi_2$ of type 1 to get an equivalent diagram Δ' with no dipoles of type 2. It is obvious that Δ and Δ' are mapped to the same homeomorphism of $[0, 1]$.

A slightly different way of describing the function in F corresponding to a given diagram Δ is the following. For each finite binary word u , we let the *interval associated with u* , denoted by $[u]$, be the dyadic interval $[.u, .u1^{\mathbb{N}}]$. If Δ is a diagram representing a homeomorphism $f \in F$, we let u_1, \dots, u_n be the positive branches of Δ and v_1, \dots, v_n be the negative branches of Δ . Then the intervals $[u_1], \dots, [u_n]$ (resp. $[v_1], \dots, [v_n]$) form a subdivision of the interval $[0, 1]$. The function f maps each interval $[u_i]$ linearly onto the interval $[v_i]$.

¹Throughout this paper, for words u and v , $u \equiv v$ denotes letter-by-letter equality.

Below, when we say that a function f has a pair of branches $u \rightarrow v$, the meaning is that some diagram representing f has this pair of branches. In other words, this is equivalent to saying that f maps $[u]$ linearly onto $[v]$. In particular, if f takes the branch u to the branch v , then for any finite binary word w , f takes the branch uw to the branch vw , where uw and vw are concatenated words. Note also that if f has a pair of branches $u \rightarrow v$ then the reduced diagram Δ of f has a pair of branches $u_1 \rightarrow v_1$ where $u \equiv u_1w$ and $v \equiv v_1w$ for some common (possibly empty) suffix w .

We will often be interested in finite dyadic fractions $\alpha \in (0, 1)$ fixed by a function $f \in F$. More generally, if $S \subset (0, 1)$, then we say that an element $f \in F$ *fixes* S , if it fixes S pointwise. The following two lemma will be useful.

Lemma 2.6. Let $f \in F$ be an element which fixes a finite dyadic fraction $\alpha \in (0, 1)$. Let $u \equiv u'1$ be a finite binary word such that $\alpha = .u$. Then the following assertions hold.

- (1) f has a pair of branches $u0^{m_1} \rightarrow u0^{m_2}$ for some $m_1, m_2 \geq 0$.
- (2) f has a pair of branches $u'01^{n_1} \rightarrow u'01^{n_2}$ for some $n_1, n_2 \geq 0$.
- (3) If $f'(\alpha^+) = 2^k$ for $k \neq 0$, then every diagram representing f has a pair of branches $u0^m \rightarrow u0^{m-k}$ for some $m \geq \max\{0, k\}$.
- (4) If $f'(\alpha^-) = 2^\ell$ for $\ell \neq 0$, then every diagram representing f has a pair of branches $u'01^n \rightarrow u'01^{n-\ell}$ for some $n \geq \max\{0, \ell\}$.

Proof. We only prove parts (1) and (3). The proof of parts (2) and (4) is analogue.

(1) Let Δ be a diagram of f such that for some $m_1, m_2 \geq 0$, $u0^{m_1}$ is a positive branch and $u0^{m_2}$ is a negative branch of Δ (such a diagram clearly exists; indeed, one can always prolong branches of Δ by inserting dipoles of type 1). Since f fixes $\alpha = .u$, it must take the branch $u0^{m_1}$ to the branch $u0^{m_2}$.

(3) Let Δ be a diagram of f . We claim that $u0^{m_1}$ must be a positive branch in Δ for some $m_1 \geq 0$. Otherwise, Δ has a positive branch u_1 where u_1 is a proper prefix of u . In that case, α belongs to the interior of $[u_1]$. Let v_1 be the negative branch of Δ such that $u_1 \rightarrow v_1$ is a pair of branches of Δ . We assume that $|u_1| \leq |v_1|$, the argument being similar in the opposite case. If u_1 is not a prefix of v_1 then $[u_1]$ and $[v_1]$ are disjoint in contradiction to f fixing α . Thus, $v_1 \equiv u_1s$. If s is empty then the function f fixes the interval $[u_1]$, in contradiction to the slope $f'(\alpha^+) \neq 1$. Otherwise, f fixes the number $.u_1s^{\mathbb{N}}$. Since f is linear on $[u_1]$, the fixed point $.u_1s^{\mathbb{N}}$ must coincide with α . Thus, $.u = .u_1s^{\mathbb{N}}$ which implies that $s \equiv 0^r$ or $s \equiv 1^r$ for some $r \in \mathbb{N}$, as $.u$ is finite dyadic. It follows that $\alpha = .u = .u_1s^{\mathbb{N}}$, in contradiction to u ending with 1 and u_1 being a proper prefix of u .

A similar argument shows that Δ must have a negative branch of the form $u0^{m_2}$ for some $m_2 \geq 0$. As in (1) it follows that Δ has a pair of branches $u0^{m_1} \rightarrow u0^{m_2}$. The assumption in (3) and the linearity of f on $[u0^{m_1}]$ implies that the slope on the interval is 2^k . Thus $m_2 = m_1 - k$, and so $m_1 \geq \max\{0, k\}$. \square

2.4 On orbitals and stabilizers in F

We will often consider the action of F on the interval $[0, 1]$. Let $\text{PL}_o(I)$ be the group of piecewise-linear orientation-preserving homeomorphisms of $[0, 1]$ with finitely many break-points. Thompson group F is clearly a subgroup of $\text{PL}_o(I)$.

Let $G \leq \text{PL}_o(I)$. The *support of G in $[0, 1]$* , denoted $\text{Supp}(G)$, is the closure of the set S of all points in $(0, 1)$, which are not fixed by G . S is a union of countably many open intervals. Each such open interval is called an orbital of G . Equivalently, an orbital of G can be defined as the convex hull of an orbit of a point x in $(0, 1)$, under the action of G , if x is not fixed by G . If $h \in \text{PL}_o(I)$, then the support of h and the orbitals of h are defined to be the support and orbitals of the group $\langle h \rangle$. Notice that an element $h \in \text{PL}_o(I)$ has finitely many orbitals and that the orbitals of h coincide with the orbitals of h^n for all $n \neq 0$.

An interval (a, b) is an orbital of h if and only if h fixes the points a and b and does not fix any number in (a, b) . The orbital (a, b) is said to be a *push-up* orbital of h , if for all $x \in (a, b)$, $f(x) > x$; equivalently, if the slope $f'(a^+) > 1$. Similarly, the orbital (a, b) is said to be a *push-down* orbital of h , if for all $x \in (a, b)$, $f(x) < x$; equivalently, if the slope $f'(a^-) < 1$. Notice that every orbital of h is either a push up or a push down orbital. If (a, b) is a push-up orbital of h then (a, b) is a push-down orbital of h^{-1} .

The following observation will be used often without a specific reference. We say that an element h has support in an interval J if the support of h is contained in J .

Remark 2.7. If $h, g \in \text{PL}_o(I)$ and (a, b) is an orbital of h , then h^g has an orbital $(g(a), g(b))$. If the support of h is contained in J_1 , then the support of h^g is contained in $g(J_1)$. If h fixes an interval J_2 then h^g fixes the interval $g(J_2)$.

2.5 The derived subgroup of F

The derived subgroup of F is a simple group [11]. It can be characterized as the subgroup of F of all functions f with slope 1 both at 0^+ and at 1^- . In other words, it is the subgroup of all functions in F with support in $(0, 1)$.

We will often be interested in subgroups of F which are not contained in any finite index subgroup of F . Since $[F, F]$ is infinite and simple, every finite index subgroup of F contains the derived subgroup of F . Thus, for a subgroup $H \leq F$ we have $H[F, F] = F$ if and only if H is not contained in any proper subgroup of finite index in F . To determine if $H[F, F] = F$, one can consider the image of F in the abelianization \mathbb{Z}^2 of F . The abelianization map $\pi_{ab}: F \rightarrow \mathbb{Z}^2$ sends x_0 to $(1, 0)$ and x_1 to $(0, 1)$. Clearly, $H[F, F] = F$ if and only if $\pi_{ab}(H) = \pi_{ab}(F) = \mathbb{Z}^2$.

Let $a < b$ be finite dyadic fractions in $(0, 1)$. We denote by $F_{[a, b]}$ the subgroup of F of all functions with support in $[a, b]$. The group $F_{[a, b]}$ is isomorphic to F . Indeed, they are conjugate subgroups of $\text{PL}_2(\mathbb{R})$, the group of all piecewise linear homeomorphisms of \mathbb{R} with finitely many breakpoints, all of which are finite dyadic fractions and where all slopes are integer powers of 2. The derived subgroup of $F_{[a, b]}$ is the subgroup of all functions with slope 1 both at a^+ and at b^- .

3 The Stallings 2-core of subgroups of diagram groups

The Stallings 2-core of a subgroup of a diagram group was defined in 1999 by Guba and Sapir and appeared first in print in [15]. The motivation was to develop a method for checking if a subgroup H of F is a strict subgroup of F (equivalently if $\{x_0, x_1\} \not\subseteq H$). This section follows [15] closely.

Recall the procedure (first discovered by Stallings [29]) of checking if an element g of a free group F_n belongs to the subgroup H generated by elements h_1, \dots, h_k . Take paths labeled by h_1, \dots, h_k . Identify the initial and terminal vertices of these paths to obtain a bouquet of circles K' with a distinguished vertex o . Then *fold* edges as follows: if there exists a vertex with two outgoing edges of the same label, we identify the edges. As a result of all the foldings (and removing the hanging trees), we obtain the *Stallings core* of the subgroup H which is a finite automaton $A(H)$ with o as its input/output vertex. Then $g \in H$ if and only if $A(H)$ accepts the reduced form of g . It is well known that the Stallings core does not depend on the generating set of the subgroup H .

In the case of diagram groups an analogue construction was given in [15]. Instead of automata we have directed 2-complexes and instead of words - diagrams.

Definition 3.1. Let $\mathcal{K} = \langle E_{\mathcal{K}} \mid \mathcal{F}_{\mathcal{K}}^+ \rangle$ be a directed 2-complex. A *2-automaton* over \mathcal{K} is a directed 2-complex $\mathcal{L} = \langle E_{\mathcal{L}} \mid \mathcal{F}_{\mathcal{L}}^+ \rangle$ with two distinguished 1-paths $p_{\mathcal{L}}$ and $q_{\mathcal{L}}$ (the input and output 1-paths), together with a map ϕ from \mathcal{L} to \mathcal{K} which takes vertices to vertices, edges to edges and cells to cells, which is a homomorphism of directed graphs and commutes with the maps **top**, **bot** and $^{-1}$. We shall call ϕ an *immersion*.

Given a diagram Δ over \mathcal{K} , we can naturally view Δ as a directed 2-complex, by considering every cell in Δ to be a pair of inverse cells. Then Δ is a 2-automaton with a natural immersion ϕ_{Δ} and the distinguished 1-paths **top**(Δ) and **bot**(Δ).

Definition 3.2. Let $\mathcal{L}, \mathcal{L}'$ be two 2-automata over \mathcal{K} . A map ψ from \mathcal{L}' to \mathcal{L} which takes vertices to vertices, edges to edges and cells to cells, which is a homomorphism of directed graphs and commutes with the maps **top**, **bot**, $^{-1}$ and the immersions is called a *morphism* from \mathcal{L}' to \mathcal{L} provided $\psi(p_{\mathcal{L}'}) = p_{\mathcal{L}}, \psi(q_{\mathcal{L}'}) = q_{\mathcal{L}}$.

Definition 3.3. We say that a 2-automaton \mathcal{L} over \mathcal{K} *accepts* a diagram Δ over \mathcal{K} if there is a morphism ψ from the 2-automaton Δ to the 2-automaton \mathcal{L} .

Let $\Delta_i, i \in \mathcal{I}$ be reduced diagrams from the diagram group $\text{DG}(\mathcal{K}, u)$ i.e., diagrams over \mathcal{K} with the same label u of their top and bottom paths. Then we can identify all **top**(Δ_i) with all **bot**(Δ_i) and obtain a 2-automaton \mathcal{L}' over \mathcal{K} with the distinguished 1-paths $p = q = \mathbf{top}(\Delta_i) = \mathbf{bot}(\Delta_i)$. We can view \mathcal{L}' as a “bouquet of spheres”. That automaton accepts any concatenation of diagrams Δ_i and their inverses.

To get a 2-automaton that accepts all reduced diagrams in the subgroup generated by $\Delta_i, i \in \mathcal{I}$, we do an analog of the Stallings foldings. Namely, let \mathcal{L}' be the 2-automaton as above. Now every time we see two cells that have the same image under the immersion of \mathcal{L}' and share the top (resp. bottom) path, then we identify their bottom (resp. top) paths and identify the cells too. This operation is called *folding of cells* (see [19, Remark 8.8]).

The result (after infinitely many foldings if \mathcal{I} is infinite) is a directed 2-complex and the immersion of \mathcal{L}' induces an immersion of the new directed 2-complex. Thus we again get a 2-automaton. Let \mathcal{L} be the 2-automaton obtained after all possible foldings were applied to \mathcal{L}' . The following 3 lemmas were proved in [15].

Lemma 3.4. The 2-automaton \mathcal{L} does not depend on the order in which foldings were applied to \mathcal{L}' .

Lemma 3.5. If the 2-automaton \mathcal{L} accepts a diagram Δ in $\text{DG}(\mathcal{K}, u)$, then it also accepts the reduced diagram equivalent to Δ . Thus, one can talk about the subgroup of $\text{DG}(\mathcal{K}, u)$

of all diagrams accepted by \mathcal{L} .

Lemma 3.6. The 2-automaton \mathcal{L} accepts all reduced diagrams from the subgroup of the diagram group $\text{DG}(\mathcal{K}, u)$ generated by Δ_i , $i \in \mathcal{I}$.

It was noted in [15] that the 2-automaton \mathcal{L} is determined uniquely by the subgroup $H = \langle \Delta_i \mid i \in \mathcal{I} \rangle$ of $\text{DG}(\mathcal{K}, u)$ and does not depend on the chosen generating set $\{\Delta_i \mid i \in \mathcal{I}\}$ (as long as all diagrams Δ_i are taken to be reduced). Thus, \mathcal{L} can be called the *Stallings 2-core* of the subgroup H . We will denote the Stallings 2-core of a subgroup H by $\mathcal{L}(H)$.

We note that unlike for subgroups of free groups, the Stallings 2-core of a subgroup H can accept diagrams not in H . Following [15], we make the following definition.

Definition 3.7. The *closure* $\text{Cl}(H)$ of a subgroup H of a diagram group $\text{DG}(\mathcal{K}, u)$ is the subgroup of $\text{DG}(\mathcal{K}, u)$ consisting of all diagrams that are accepted by the 2-core $\mathcal{L}(H)$ of H . If $H = \text{Cl}(H)$ we say that H is a *closed* subgroup of $\text{DG}(\mathcal{K}, u)$.

It is clear that all usual conditions of the closure operation are satisfied, that is, $H \leq \text{Cl}(H)$ (Lemma 3.6), $\text{Cl}(\text{Cl}(H)) = \text{Cl}(H)$ (indeed, the core $\mathcal{L}(\text{Cl}(H))$ coincides with the core of H) and if $H_1 \leq H_2$, then $\text{Cl}(H_1) \leq \text{Cl}(H_2)$. If H is finitely generated, then the core $\mathcal{L}(H)$ is finite, and so the membership problem in $\text{Cl}(H)$ is decidable.

We make the following simple observation.

Lemma 3.8. Let H be a subgroup of a diagram group $\text{DG}(\mathcal{K}, u)$. Let $\mathcal{L}(H)$ be the core of H and let $p = p_{\mathcal{L}(H)}$ be the distinguished 1-path of $\mathcal{L}(H)$. The closure $\text{Cl}(H)$ is naturally isomorphic to the diagram group $\text{DG}(\mathcal{L}(H), p)$, where $\mathcal{L}(H)$ is viewed as a directed 2-complex.

Proof. The immersion ϕ from $\mathcal{L}(H)$ to \mathcal{K} , enables to view any diagram over $\mathcal{L}(H)$ as a diagram over \mathcal{K} . Indeed, if Δ is a diagram over $\mathcal{L}(H)$, then every edge (resp. cell) of Δ is labeled by an edge e (resp. cell f) of $\mathcal{L}(H)$. One can relabel it by the edge $\phi(e)$ (resp. the cell $\phi(f)$) of \mathcal{K} . In particular, every diagram in $\text{DG}(\mathcal{L}(H), p)$ can be viewed as a diagram over \mathcal{K} which is obviously accepted by $\mathcal{L}(H)$, hence belongs to $\text{Cl}(H)$. We claim that this mapping ψ from $\text{DG}(\mathcal{L}(H), p)$ to $\text{Cl}(H)$ is an isomorphism. It is easy to see that the mapping is a homomorphism. To prove injectivity, let Δ be a reduced diagram in $\text{DG}(\mathcal{L}, p)$ and let Δ' be its image in $\text{Cl}(H)$. Suppose that cells π'_1 and π'^{-1}_2 form a dipole in Δ' and let π_1 and π_2 be the cells of Δ which map onto π'_1 and π'_2 . Let f_1 and f_2 be the labels of the cells π_1 and π_2 in $\mathcal{L}(H)$. Since $\text{bot}(\pi_1) = \text{bot}(\pi_2)$, we have $\text{bot}(f_1) = \text{bot}(f_2)$. Since $\pi'_1 \circ \pi'^{-1}_2$ is a dipole in Δ' , we have $\phi(f_1) = \phi(f_2)$. As no foldings are applicable to $\mathcal{L}(H)$, the cells f_1 and f_2 must coincide. Hence, π_1 and π_2^{-1} have mutually inverse labels and so $\pi_1 \circ \pi_2^{-1}$ forms a dipole in Δ , in contradiction to Δ being reduced. To prove that the mapping ψ is onto, we observe that if Δ' is a diagram in $\text{Cl}(H)$, then the morphism from Δ' to $\mathcal{L}(H)$, enables to view it as a diagram in $\text{DG}(\mathcal{L}, p)$, which maps to Δ' by the homomorphism ψ . \square

We demonstrate the construction of the Stallings 2-core of the subgroup $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$ of Thompson group F and demonstrate how to check if an element $f \in F$ belongs to $\text{Cl}(H)$. Let us denote the positive cell of the Dunce hat $\langle x \mid x \rightarrow x^2 \rangle$ by π . The diagrams for x_0 and $x_1 x_2 x_1^{-1}$ viewed as 2-automata are in Figure 3.5 below (the immersion to the Dunce hat maps all positive cells to π , and all edges to the only edge of the Dunce hat).

Together the two diagrams have 20 edges (labeled by e_1, \dots, e_{20}) and 12 cells (labeled $\gamma_1, \dots, \gamma_{12}$). To construct the 2-automaton \mathcal{L} for these two diagrams, we first need to identify

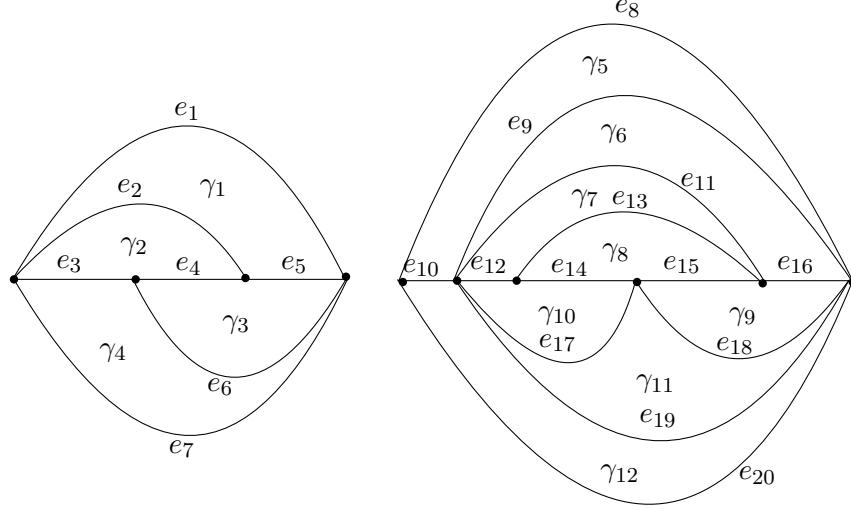


Figure 3.5: The diagrams for x_0 and $x_1x_2x_1^{-1}$

the top and the bottom paths of both diagrams. So we set $e_1 = e_7 = e_8 = e_{20}$. Now the positive cells $\gamma_1, \gamma_4, \gamma_5, \gamma_{12}$ need to be folded because these cells share the top path e_1 . So we need to identify $\gamma_1 = \gamma_4 = \gamma_5 = \gamma_{12}$ and the edges $e_2 = e_3 = e_{10}$ and $e_5 = e_6 = e_9 = e_{19}$. Now the cells $\gamma_3, \gamma_6, \gamma_{11}$ have common top edge e_5 . So we need to fold these three cells. Thus $\gamma_3 = \gamma_6 = \gamma_{11}$, $e_4 = e_{11} = e_{17}$, $e_5 = e_{16} = e_{18}$. Then the cells γ_7 and γ_{10} share the top edge e_4 . So we set $\gamma_7 = \gamma_{10}$, $e_{13} = e_{14}$. Furthermore γ_9 and γ_3 now share the top edge e_5 . So we need to set $e_4 = e_{15}$. No more foldings are needed, and the 2-automaton \mathcal{L} is presented in Figure 3.6 (there the cells and edges are supposed to be identified according to their labels: all e_1 edges are the same, all γ_7 -cells are the same, etc.).

Now consider the element x_1 . The diagram Δ for x_1 with labels of edges and cells is in Figure 3.7. If $x_1 \in \text{Cl}(H)$, then we should have a morphism ψ from Δ to \mathcal{L} sending f_1 and f_{10} to e_1 . Then $\psi(\delta_1) = \psi(\delta_6) = \gamma_1$ since \mathcal{L} has only one cell with top edge e_1 . This forces $\psi(f_2) = e_2$, $\psi(f_3) = \psi(f_9) = e_5$. Since \mathcal{L} has only one positive cell with top edge e_5 , we should have $\psi(\delta_2) = \gamma_3$. That means $\psi(f_4) = e_4, \psi(f_5) = e_5$. Again \mathcal{L} has only one positive cell with top edge e_4 . Therefore $\psi(\delta_3) = \gamma_7$, hence $\psi(f_6) = e_{12}, \psi(f_7) = e_{13}$. Now ψ must map the positive cell δ_4 to a cell with bottom edges $\psi(f_7) = e_{13}$ and $\psi(f_5) = e_5$. But \mathcal{L} does not have such a cell, a contradiction. Thus, $x_1 \notin \text{Cl}(H)$.

Now, let \mathcal{K} be a directed 2-complex and let $\text{DG}(\mathcal{K}, u)$ be a diagram group over \mathcal{K} . We say that a 2-automaton over \mathcal{K} is *folded* if no foldings are applicable to it. The following is a simple observation.

Definition 3.9. Let Δ be a diagram in $\text{DG}(\mathcal{K}, u)$. Assume that there are diagrams Ψ, Δ_1 and Δ_2 in the diagram groupoid $\mathcal{D}(\mathcal{K})$ such that Ψ is a (vw, u) -diagram, Δ_1 is a (v, v) -diagram and Δ_2 is a (w, w) -diagram. Assume also that $\Delta \equiv \Psi^{-1} \circ (\Delta_1 + \Delta_2) \circ \Psi$ (see Figure

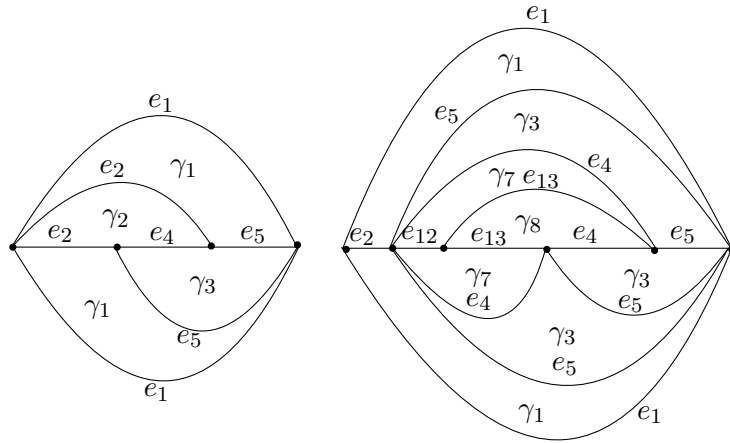


Figure 3.6: The 2-automaton for $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$

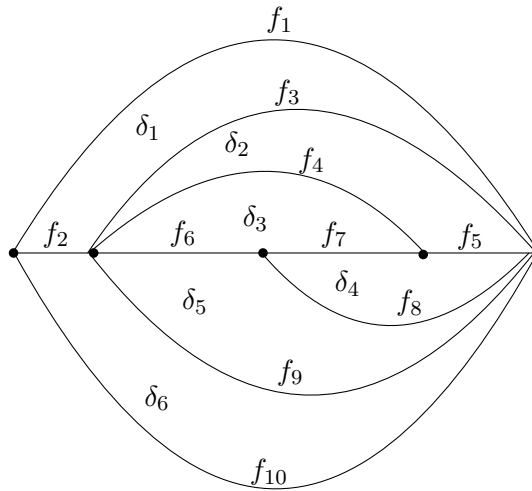


Figure 3.7: The 2-automaton for x_1

3.8). Then the diagrams

$$\Psi^{-1} \circ (\Delta_1 + \varepsilon(w)) \circ \Psi \quad \text{and} \quad \Psi^{-1} \circ (\varepsilon(v) + \Delta_2) \circ \Psi$$

are called *components* of the diagram Δ . (Notice that a diagram Δ can have more than 2 components.)

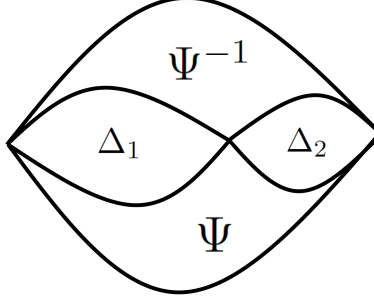


Figure 3.8: $\Delta \equiv \Psi^{-1} \circ (\Delta_1 + \Delta_2) \circ \Psi$

Remark 3.10 (Guba and Sapir). Let \mathcal{L} be a folded 2-automaton over the directed 2-complex \mathcal{K} with distinguished 1-path $p_{\mathcal{L}} = q_{\mathcal{L}}$. Let Δ be a diagram in $\text{DG}(\mathcal{K}, u)$. If Δ is accepted by \mathcal{L} then all components of Δ are also accepted by \mathcal{L} . In particular, if $H \leq \text{DG}(\mathcal{K}, u)$, and $\Delta \in \text{Cl}(H)$, then all components of Δ also belong to $\text{Cl}(H)$. We say that $\text{Cl}(H)$ is *closed for components*.

The following conjecture is due to Guba and Sapir. The conjecture was made around 1999 but was never formulated in print.

Conjecture 3.11 (Guba and Sapir). Let H be a subgroup of a diagram group $\text{DG}(\mathcal{K}, u)$. Then the closure $\text{Cl}(H)$ is the minimal subgroup of $\text{DG}(\mathcal{K}, u)$ which contains H and is closed for components.

In Section 5 we prove Conjecture 3.11 for subgroups of Thompson group F . The conjecture for general diagram groups remains open.

4 Paths on the core of a subgroup $H \leq F$

Let \mathcal{K} be the Duncce hat. From now on, all 2-automata \mathcal{L} considered in this paper are 2-automata over \mathcal{K} , unless explicitly stated otherwise. Positive cells of \mathcal{L} are mapped by the immersion $\phi_{\mathcal{L}}$ to the positive cell of the Duncce hat. Other than diagrams Δ in F which are sometimes viewed as 2-automata, we will only consider 2-automata \mathcal{L} where the distinguished 1-paths $p_{\mathcal{L}}$ and $q_{\mathcal{L}}$ coincide and are composed of a single edge e so that e is mapped by $\phi_{\mathcal{L}}$ to the unique edge of the Duncce hat. We assume that all 2-automata \mathcal{L} below satisfy these properties, even if it is not mentioned explicitly. Since every positive cell π in \mathcal{L} is mapped to the positive cell of \mathcal{K} , the top path $\mathbf{top}(\pi)$ is composed of one edge and the bottom path $\mathbf{bot}(\pi)$ is composed of two (left and right) edges.

We will need to distinguish between two types of foldings applicable to a 2-automaton \mathcal{L} over \mathcal{K} . If two positive cells π_1 and π_2 of \mathcal{L} share their top paths but not their bottom paths, then folding π_1 and π_2 (and their inverse cells) is considered a folding of *type 1*. If π_1 and π_2 share their bottom paths, then a folding of π_1 and π_2 is a folding of *type 2*.

We define paths on a 2-automaton \mathcal{L} in a similar way to the definition of paths on tree-diagrams over \mathcal{K} (see Section 2.3).

Definition 4.1. Let \mathcal{L} be a 2-automaton over \mathcal{K} with distinguished edge $p_{\mathcal{L}} = q_{\mathcal{L}}$. A finite sequence of edges e_1, \dots, e_n in the 2-automaton \mathcal{L} is said to be a *path* on \mathcal{L} if

- (1) $e_1 = p_{\mathcal{L}} = q_{\mathcal{L}}$; and
- (2) for each $i = 1, \dots, n-1$, the edge e_i is the top edge of some positive cell π_i in \mathcal{L} and e_{i+1} is a bottom left or right edge of the same cell.

The label $\text{lab}(p)$ of a path $p = e_1, \dots, e_n$ on \mathcal{L} is defined in the same way as the label of a path on a tree-diagram Ψ (see Definition 2.4).

Note that if \mathcal{L} is a 2-automaton to which no foldings of type 1 are applicable then a path p on \mathcal{L} is uniquely determined by its label. In that case, we will often abuse notation and refer to a path in terms of its label. In particular, given a finite binary word u , we could refer to a path u on \mathcal{L} and to a path u on a 2-automaton \mathcal{L}' (where no foldings of type 1 are applicable) at the same time. We can also refer to u as a (positive or negative) path on a diagram Δ . This should not cause any confusion as we are careful to mention the 2-automaton or diagram we are referring to.

Note that if \mathcal{L} is a 2-automaton and u is a finite binary word then u does not necessarily label a path on \mathcal{L} . If every edge in \mathcal{L} is the top edge of some positive cell, then every binary word u labels (at least one) path on \mathcal{L} . If p is a path on the 2-automaton \mathcal{L} such that $\text{lab}(p) \equiv u$, then p^+ and u^+ denote the last edge of the path in \mathcal{L} .

The following remarks would often be used below with no specific reference.

Remark 4.2. Let \mathcal{L} be a 2-automaton over \mathcal{K} with distinguished edge $p_{\mathcal{L}} = q_{\mathcal{L}}$. Let v be a finite binary word. Then $v0$ labels a path on \mathcal{L} if and only if $v1$ labels a path on \mathcal{L} . Thus, if a word u labels a path on \mathcal{L} and Ψ is the minimal tree-diagram over \mathcal{K} with branch u then every branch b of Ψ labels a path on \mathcal{L} .

Remark 4.3. Let \mathcal{L}' and \mathcal{L} be 2-automata over the Duncce hat \mathcal{K} such that $p_{\mathcal{L}'} = q_{\mathcal{L}'}$ and $p_{\mathcal{L}} = q_{\mathcal{L}}$ are composed of a single edge. A morphism ψ from \mathcal{L}' to \mathcal{L} naturally sends every path on the automaton \mathcal{L}' to a path on \mathcal{L} with the same label.

Lemma 4.4. Let \mathcal{L}' and \mathcal{L} be 2-automata over \mathcal{K} such that \mathcal{L} results from \mathcal{L}' by applications of foldings of type 2. Then any path on \mathcal{L} can be lifted to a unique path on \mathcal{L}' . In particular, by Remark 4.3, there is a 1 – 1 correspondence between paths on \mathcal{L}' and paths on \mathcal{L} .

Proof. We can assume that \mathcal{L} results from \mathcal{L}' by an application of a single folding of type 2. Clearly, that induces a morphism ψ from \mathcal{L}' to \mathcal{L} , so by Remark 4.3, any path on \mathcal{L}' is mapped to a path on \mathcal{L} . Let π_1 and π_2 be the positive cells of \mathcal{L}' which are folded in the transition to \mathcal{L} . In particular, in \mathcal{L}' , $\text{bot}(\pi_1) = \text{bot}(\pi_2)$; the cells π_1 and π_2 become identified in \mathcal{L} (i.e., $\psi(\pi_1) = \psi(\pi_2)$) and the top edges $\text{top}(\pi_1)$ and $\text{top}(\pi_2)$ are folded to a single edge e of \mathcal{L} . Let $p = e_1, \dots, e_n$ be a path on \mathcal{L} . We prove by induction on n that p can be lifted to a unique path $q = e'_1, \dots, e'_n$ on \mathcal{L}' such that for all i , $\psi(e'_i) = e_i$. If $n = 1$, the result is clear. Assume that the lemma holds for n and let $p = e_1, \dots, e_n, e_{n+1}$. By assumption, the path e_1, \dots, e_n can be lifted to a unique path e'_1, \dots, e'_n . Notice that e_n must be the top edge of some positive cell π in \mathcal{L} such that e_{n+1} is a bottom edge of π . If π is not the folded cell, i.e., π is not the cell $\psi(\pi_1) = \psi(\pi_2)$, then there is a unique cell π' in \mathcal{L}' such that $\pi = \psi(\pi')$. Then e'_n is the top edge of π' . If e_{n+1} is the left (resp. right) bottom edge of π , then one should take e'_{n+1} to be the left (resp. right) bottom edge of π' .

That would complete the lifting of the path p and it is obviously the only choice for e'_{n+1} . If π is the folded cell, then $e'_n = \mathbf{top}(\pi_1)$ or $e'_n = \mathbf{top}(\pi_2)$. In that case, if e_{n+1} is a left (resp. right) bottom edge of π , one should take e'_{n+1} to be the common left (resp. right) bottom edge of π_1 and π_2 . \square

For the proof of Lemma 4.6 below we will have to consider paths on a 2-automaton \mathcal{L} which do not start from the distinguished edge $p_{\mathcal{L}} = q_{\mathcal{L}}$. A sequence p of edges e_1, \dots, e_n in \mathcal{L} is called a *trail* if it satisfies the second condition in Definition 4.1. If p_1 and p_2 are trails such that the terminal edge of p_1 is the initial edge of p_2 , then the concatenation of trails is naturally defined. We denote the concatenation of p_1 and p_2 by $p_1 p_2$. Note that if $p = e_1, \dots, e_n$ is a path on \mathcal{L} then it can be viewed as a concatenation of $n - 1$ trails p_1, \dots, p_{n-1} where for each i , $p_i = e_i, e_{i+1}$. In particular we get the following.

Lemma 4.5. Let \mathcal{L}' and \mathcal{L} be 2-automata over \mathcal{K} such that \mathcal{L}' projects onto \mathcal{L} . Let p be a path on \mathcal{L} . Then p is a concatenation of trails p_1, \dots, p_n such that each trail p_i can be lifted to a trail on \mathcal{L}' . \square

Let H be a subgroup of F . We consider paths on the core $\mathcal{L}(H)$. Note that if e is an edge of $\mathcal{L}(H)$, then there is a path p on $\mathcal{L}(H)$ such that $p^+ = e$. Indeed, this is already true for the bouquet of spheres \mathcal{L}' defined in the construction of $\mathcal{L}(H)$ and \mathcal{L}' projects onto $\mathcal{L}(H)$.

Lemma 4.6. Let H be a subgroup of F and let $\mathcal{L}(H)$ be the core of H . Let p and q be two paths on the core $\mathcal{L}(H)$ with labels $\text{lab}(p) \equiv u$ and $\text{lab}(q) \equiv v$. Assume that $p^+ = q^+$ (i.e., the paths p and q terminate on the same edge of $\mathcal{L}(H)$). Then there is an integer $k \geq 0$, such that for all finite binary word w of length $\geq k$, there is an element $h \in H$ with a pair of branches $uw \rightarrow vw$.

Proof. We consider the construction of the core $\mathcal{L}(H)$. Let $\{\Delta_i \mid i \in \mathcal{I}\}$ be a generating set of H . It is enough to consider the case where \mathcal{I} is infinite. The first step in the construction of $\mathcal{L}(H)$ is to identify all the top and bottom edges of the generators Δ_i to get a 2-automaton $\mathcal{L}' = \mathcal{L}'_0$. Next, we apply countably many foldings to \mathcal{L}' , so that if \mathcal{L}'_n , $n \in \mathbb{N}$ are the 2-automata resulting in the process, then no folding is applicable to the limit automaton $\mathcal{L} = \mathcal{L}(H)$.

It is enough to prove that the lemma holds for each of the automata \mathcal{L}'_n , $n \geq 0$. Indeed, if p and q are paths on the core $\mathcal{L}(H)$ such that $p^+ = q^+$, then for a large enough $n \in \mathbb{N}$ they can be lifted to paths p_n, q_n on \mathcal{L}'_n such that $p_n^+ = q_n^+$ and $\text{lab}(p_n) \equiv \text{lab}(p)$, $\text{lab}(q_n) \equiv \text{lab}(q)$.

We make the following claim. For each $n \geq 0$, let k_n be the number of foldings of type 2 out of the n foldings applied to \mathcal{L}' to get the 2-automaton \mathcal{L}'_n . Then the lemma holds for any pair of paths p and q on the 2-automaton \mathcal{L}'_n with the constant $k = k_n$.

We prove the claim by induction on n . For $n = 0$, let p and q be paths on \mathcal{L}'_0 such that $p^+ = q^+ = e$. Recall that \mathcal{L}'_0 is a bouquet of the diagrams Δ_i (each, with the top and bottom edges identified). If e is not an edge on the horizontal 1-path of any of the diagrams Δ_i , then the paths p and q , and their labels u and v , must coincide. Then for any word w of length $\geq k_0 = 0$, the identity element of H has the pair of branches $uw \rightarrow vw$. If e lies on the horizontal 1-path of some Δ_i and p and q do not coincide, then $u \rightarrow v$ is a pair of branches of the diagram Δ_i or its inverse. In particular, for every binary word w of length $\geq k_0 = 0$, a diagram equivalent to Δ_i or Δ_i^{-1} has the pair of branches $uw \rightarrow vw$.

Let $n \in \mathbb{N}$ and assume that the claim holds for $n - 1$. We consider two cases.

Case 1: The n^{th} folding is a folding of type 1. In that case $k_n = k_{n-1}$.

Let p and q be paths on \mathcal{L}'_n such that $p^+ = q^+$. By Lemma 4.5, p (resp. q) is a concatenation of trails p_1, \dots, p_m (resp. q_1, \dots, q_r) which can be lifted to trails on \mathcal{L}'_{n-1} . We prove the claim by induction on $m + r$ (when m and r are taken to be the smallest possible for p and q). Assume first that $m = 1$ and $r = 1$.

Let p' and q' be liftings of p and q to paths on \mathcal{L}'_{n-1} . If $(p')^+ = (q')^+$ then we are done by the induction hypothesis. Otherwise, the edges $(p')^+$ and $(q')^+$ of \mathcal{L}'_{n-1} are identified in \mathcal{L}'_n as a result of the unique folding of type 1 applied to \mathcal{L}'_{n-1} . It follows that there are two positive cells π_1 and π_2 in \mathcal{L}'_{n-1} , such that $\mathbf{top}(\pi_1) = \mathbf{top}(\pi_2)$ and such that $(p')^+$ is the left or right bottom edge of π_1 and $(q')^+$ is the respective bottom edge of π_2 . We assume that $(p')^+$ is a left bottom edge, the other case being similar. It follows that $u \equiv \text{lab}(p') \equiv u'0$ and $v \equiv \text{lab}(q') \equiv v'0$ for some prefixes u' and v' . Let p'' and q'' be the initial subpaths of p' and q' labeled by the prefixes u' and v' . It is obvious that $(p'')^+ = (q'')^+ = \mathbf{top}(\pi_1)$. By the induction hypothesis, for any finite binary word w' of length $\geq k_{n-1}$ there is an element $h \in H$ with a pair of branches $u'w' \rightarrow v'w'$. For any word w of length $\geq k_n = k_{n-1}$, one can take $w' \equiv 0w$. Then there is an element $h \in H$ with a pair of branches $u'0w \equiv uw \rightarrow v'0w \equiv vw$.

If $m > 1$ and $p = p_1 \cdots p_m$, we let p'_i , $i = 1, \dots, m$ be the lifting of the trail p_i to \mathcal{L}'_{n-1} . Let e be the initial edge of p'_2 . There is a path p'_1 on \mathcal{L}'_{n-1} with terminal edge e . Let s be the projection of p'_1 to a path on \mathcal{L}'_n . Then p_1 and s terminate on the same edge. By the case $m = r = 1$, we have that for any word w' of length $\geq k_n$ there is an element h_1 in H with a pair of branches $\text{lab}(p_1)w' \rightarrow \text{lab}(s)w'$. We consider the path $sp_2p_3 \cdots p_m$ on \mathcal{L}'_n . Note that sp_2 can be lifted to the path $p'_1p'_2$ on \mathcal{L}'_{n-1} , thus $sp_2p_3 \cdots p_m$ is a concatenation of $m - 1$ trails such that each one can be lifted to a trail on \mathcal{L}'_{n-1} . By the induction hypothesis, for each w of length $\geq k_n$ there is an element h_2 in H with a pair of branches $\text{lab}(s)\text{lab}(p_2 \cdots p_m)w \rightarrow \text{lab}(q)w$. Then if one takes $w' \equiv \text{lab}(p_2 \cdots p_m)w$, then $|w'| \geq k_n$ and from the above there exists $h_1 \in H$ with a pair of branches $\text{lab}(p_1)\text{lab}(p_2 \cdots p_m)w \rightarrow \text{lab}(s)\text{lab}(p_2 \cdots p_m)w$. Then h_1h_2 is an element of H with a pair of branches $\text{lab}(p)w \rightarrow \text{lab}(q)w$, as required. The argument for $r > 1$ is similar.

Case 2: The n^{th} folding is a folding of type 2. In that case, $k_n = k_{n-1} + 1$.

Let p and q be paths on \mathcal{L}'_n such that $p^+ = q^+$. Let p_1 and q_1 be liftings of p and q to paths on \mathcal{L}'_{n-1} (see Lemma 4.4). As in case (1), we only have to consider the case where p_1^+ and q_1^+ are distinct edges in \mathcal{L}'_{n-1} which are identified in \mathcal{L}'_n as a result of the folding applied to \mathcal{L}'_{n-1} . Since the folding is of type 2, there are two positive cells π_1 and π_2 in \mathcal{L}'_{n-1} such that $\mathbf{bot}(\pi_1) = \mathbf{bot}(\pi_2)$, the edge $p_1^+ = \mathbf{top}(\pi_1)$ and the edge $q_1^+ = \mathbf{top}(\pi_2)$. Let w be a word of length $\geq k_n$. Then $w \equiv aw'$ where $a \in \{0, 1\}$ and $|w'| \geq k_{n-1}$. The path p_1 can be extended to a path p'_1 in \mathcal{L}'_{n-1} such that $\text{lab}(p'_1) \equiv \text{lab}(p_1)a$ and the terminal edge $(p'_1)^+$ is a bottom edge of π_1 (it is the left bottom edge if $a \equiv 0$ and the right bottom edge if $a \equiv 1$). Similarly, the path q_1 can be extended to a path q'_1 in \mathcal{L}'_{n-1} such that $\text{lab}(q'_1) \equiv \text{lab}(q_1)a$ and the terminal edge $(q'_1)^+$ is a bottom edge of π_2 . Clearly, in \mathcal{L}'_{n-1} , the edges $(p'_1)^+$ and $(q'_1)^+$ coincide. Since $|w'| \geq k_{n-1}$, by the induction hypothesis, there is an element $h \in H$ with a pair of branches $\text{lab}(p'_1)w' \rightarrow \text{lab}(q'_1)w'$. Since $\text{lab}(p'_1)w' \equiv \text{lab}(p)w$ and $\text{lab}(q'_1)w' \equiv \text{lab}(q)w$, h has a pair of branches $\text{lab}(p)w \rightarrow \text{lab}(q)w$, as required. \square

Remark 4.7. The proof of Lemma 4.6 implies that if H is a finitely generated subgroup of

F then there is a uniform constant $k \in \mathbb{N}$ such that for any two paths u and v on the core $\mathcal{L}(H)$ and for any finite binary word w of length $\geq k$, if $u^+ = v^+$ then there is an element $h \in H$ with a pair of branches $uw \rightarrow vw$.

The following simple lemma can be seen as a partial converse to Lemma 4.6.

Lemma 4.8. Let $H \leq F$ be a subgroup of F . If $h \in H$ has a pair of branches $u \rightarrow v$ such that u and v label paths on the core $\mathcal{L}(H)$, then the terminal edges u^+ and v^+ coincide. In particular, if Δ is a reduced diagram in H then for any pair of branches $w_1 \rightarrow w_2$ of Δ , we have $w_1^+ = w_2^+$ in $\mathcal{L}(H)$.

Proof. Let Δ_1 be the reduced diagram of h . Then Δ_1 has a pair of branches $u_1 \rightarrow v_1$ such that $u \equiv u_1 w$ and $v \equiv v_1 w$ for some common suffix w . Clearly, it suffices to prove that the terminal edges u_1^+ and v_1^+ coincide in $\mathcal{L}(H)$. The natural morphism from Δ_1 to the core $\mathcal{L}(H)$, maps the branches u_1 and v_1 of Δ_1 to paths p_1 and q_1 on $\mathcal{L}(H)$ such that $\text{lab}(p_1) \equiv u_1$ and $\text{lab}(q_1) \equiv v_1$. Since the terminal edges of the branches u_1 and v_1 coincide in Δ_1 , the terminal edges $p_1^+ = u_1^+$ and $q_1^+ = v_1^+$ coincide in $\mathcal{L}(H)$.

For the last statement of the lemma, notice that if Δ is reduced, then w_1 and w_2 must label paths on $\mathcal{L}(H)$ since Δ is accepted by $\mathcal{L}(H)$. \square

Lemma 4.6 implies the following.

Corollary 4.9. Let H be a subgroup of F . Consider the finite binary words $u \equiv \emptyset$, $v \equiv 0^m$ and $w \equiv 1^n$ for $m, n \in \mathbb{N}$. Let s be a finite binary word which contains both digits 0 and 1 and assume that u, v, w and s label paths on the core $\mathcal{L}(H)$ (the empty path can always be considered as a path on $\mathcal{L}(H)$). Then the terminal edges u^+ , v^+ , w^+ and s^+ on $\mathcal{L}(H)$ are all distinct edges.

Proof. If $u^+ = v^+$, then by Lemma 4.6 there is an integer $k \in \mathbb{N}$ such that for every finite binary word r of length $\geq k$, there is an element $h \in H$ with a pair of branches $ur \rightarrow vr$. Let $r \equiv 1^k$. Then there is an element $h \in H$ with a pair of branches $ur \equiv 1^k \rightarrow vr \equiv 0^m 1^k$. Then h maps $1 = .1^{\mathbb{N}}$ onto $.0^m 1^{\mathbb{N}} = .0^{m-1} 1 \neq 1$, in contradiction to h being a homeomorphism of $[0, 1]$. The proof for the other cases is similar. \square

Now let $H \leq F$ and let $\mathcal{L}(H)$ be the core of H . Let $p_{\mathcal{L}(H)} = q_{\mathcal{L}(H)}$ be the distinguished 1-path of $\mathcal{L}(H)$. We denote by $\iota(\mathcal{L}(H))$ the initial vertex of $p_{\mathcal{L}(H)}$ and by $\tau(\mathcal{L}(H))$ the terminal vertex of $p_{\mathcal{L}(H)}$. The vertex $\iota(\mathcal{L}(H))$ is called the *initial vertex* of $\mathcal{L}(H)$ and $\tau(\mathcal{L}(H))$ is the *terminal vertex* of the core $\mathcal{L}(H)$. Any other vertex of $\mathcal{L}(H)$ is an *inner vertex*. An edge of $\mathcal{L}(H)$ is an *inner edge* if both of its endpoints are inner vertices. Note that every inner vertex in $\mathcal{L}(H)$ has at least one incoming and one outgoing edge. $\iota(\mathcal{L}(H))$ has only outgoing edges and $\tau(\mathcal{L}(H))$ has only incoming edges.

As noted above, for any edge e in $\mathcal{L}(H)$ there is a path u on $\mathcal{L}(H)$ such that $u^+ = e$. It is easy to see (or prove by induction) that if e is incident to $\iota(\mathcal{L}(H))$ then u can be taken to be of the form $u \equiv 0^m$ for $m \geq 0$. Conversely, if $u \equiv 0^k$ for some $k \geq 0$ then u^+ is incident to $\iota(\mathcal{L}(H))$. Similarly, if e is incident to $\tau(\mathcal{L}(H))$ then there is a path $u \equiv 1^n$ for $n \geq 0$ such that $u^+ = e$ and if $u \equiv 1^r$ for some $r \geq 0$ then u^+ is incident to $\tau(\mathcal{L}(H))$. Corollary 4.9 implies the following.

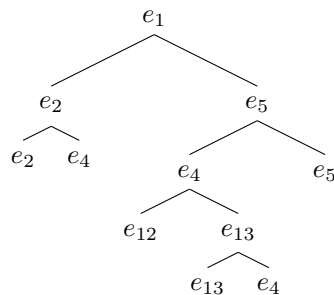
Corollary 4.10. Let $H \leq F$. Let u be a finite binary word which labels a path on $\mathcal{L}(H)$. Then

- (1) u contains the digit 0 if and only if u^+ is not incident to $\tau(\mathcal{L}(H))$.
- (2) u contains the digit 1 if and only if u^+ is not incident to $\iota(\mathcal{L}(H))$.
- (3) u contains both digits 0 and 1 if and only if u^+ is an inner edge of $\mathcal{L}(H)$.

An edge of $\mathcal{L}(H)$ is called a *boundary edge* if it is not an inner edge. If it is incident to $\iota(\mathcal{L}(H))$ (resp. $\tau(\mathcal{L}(H))$), but is not the distinguished edge of $\mathcal{L}(H)$, then it is a *left* (resp. *right*) boundary edge.

Remark 4.11. Given a subgroup $H \leq F$, we describe the core $\mathcal{L}(H)$ as a 2-automaton by listing its edges, listing its positive cells and noting what the distinguished edge is. Moreover, to describe a positive cell of $\mathcal{L}(H)$ uniquely it is enough to note the labels of its top and bottom paths.

It is often convenient to describe $\mathcal{L}(H)$ using a labeled binary tree T , where every vertex is labeled by an edge of $\mathcal{L}(H)$; the root is labeled by the distinguished edge $p_{\mathcal{L}(H)} = q_{\mathcal{L}(H)}$ and every caret in T corresponds to a positive cell of $\mathcal{L}(H)$ and vice versa. For example, the core $\mathcal{L}(H)$ for $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$, constructed in Section 3, can be described by the following binary tree.



The distinguished edge of $\mathcal{L}(H)$ is e_1 , the inner edges of $\mathcal{L}(H)$ are e_4 , e_{12} and e_{13} . The edge e_2 is a left boundary edge and e_5 is a right boundary edge. We note that even though vertices are not listed in the tree we can tell from the tree that there are 2 inner vertices in $\mathcal{L}(H)$: $\iota(e_5) = \iota(e_4) = \iota(e_{12})$ and $\iota(e_{13})$. The vertices of $\mathcal{L}(H)$ will be discussed more in detail in Section 6.

5 The closure of a subgroup $H \leq F$

In this section we prove Conjecture 3.11 for the closure of subgroups of Thompson group F . First we give an equivalent definition for components of an element of F .

Definition 5.1. Let f be a function in F . If f fixes a finite dyadic fraction $\alpha \in (0, 1)$, then the following functions $f_1, f_2 \in F$ are called *components* of the function f , or *components of f at α* .

$$f_1(t) = \begin{cases} f(t) & \text{if } t \in [0, \alpha] \\ t & \text{if } t \in [\alpha, 1] \end{cases} \quad f_2(t) = \begin{cases} t & \text{if } t \in [0, \alpha] \\ f(t) & \text{if } t \in [\alpha, 1] \end{cases}$$

Note that a function $f \in F$ can have more than two components. Indeed, f can fix more than one finite dyadic fraction. We claim that Definition 5.1 is equivalent to Definition 3.9. Clearly, it is enough to consider components of reduced diagrams Δ in F .

Lemma 5.2. Let $f \in F$ be an element represented by a reduced diagram Δ . Then a function $g \in F$ is a component of the function f , as in Definition 5.1, if and only if it is represented by a diagram Δ' which is a component of Δ , as in Definition 3.9.

Proof. We first consider components of the diagram Δ . Assume that there are diagrams Ψ, Δ_1 and Δ_2 in the diagram groupoid $\mathcal{D}(\mathcal{K})$ (where \mathcal{K} is the Dunce hat), such that Δ_1 is a spherical (x^n, x^n) -diagram, Δ_2 is a spherical (x^m, x^m) -diagram, Ψ is an (x^{m+n}, x) -diagram and such that

$$\Delta \equiv \Psi^{-1} \circ (\Delta_1 + \Delta_2) \circ \Psi.$$

We consider the diagrams $\Psi^{\pm 1}, \Delta_1$ and Δ_2 as subdiagrams of Δ . Let e be the first edge of $\mathbf{top}(\Delta_2)$ (see Figure 5.9). We claim that e must be an edge of the positive subdiagram Δ^+ . Indeed, if e is not an edge of Δ^+ , then e must be the bottom edge of an (x^2, x) -cell π which belongs to Ψ^{-1} . (Indeed, every edge of Δ which is not an edge of Δ^+ is the bottom edge of some (x^2, x) -cell.) The edge e is also the $(n+1)$ -edge of $\mathbf{bot}(\Psi^{-1})$. Let e' be the corresponding edge of Ψ . That is, e' is the $(n+1)$ -edge in $\mathbf{top}(\Psi)$, when Ψ is viewed as a subdiagram of Δ . Since Δ_1 is an (x^n, x^n) -diagram, the edge e' is the first edge of $\mathbf{bot}(\Delta_2)$. Clearly, the edge e' is the top edge of an (x, x^2) -cell of Ψ , corresponding to the cell π of Ψ^{-1} . Therefore, e' belongs to the positive subdiagram Δ^+ of Δ . That contradicts the assumption that e is not an edge of Δ^+ , as e lies above e' . A similar argument shows that the first edge e' of $\mathbf{bot}(\Delta_2)$ is an edge of the negative subdiagram Δ^- .

Let u be the left most positive branch of Δ which visits the edge e and let v be the left most negative branch of Δ which visits the edge e' . It is obvious that $u \rightarrow v$ is a pair of branches of the diagram Δ . Indeed, u^+ and v^+ are the left most edge on the horizontal 1-path of the subdiagram Δ_2 of Δ . Notice that u has a prefix u_1 which is a branch of ψ^{-1} (with terminal edge e) and that $u \equiv u_1 0^{k_1}$ for some k_1 . Similarly, v has an initial subpath v_1 such that $v \equiv v_1 0^{k_2}$ for some k_2 and such that $v_1^+ = e'$ (see Figure 5.9). Since e and e' are corresponding edges of Ψ^{-1} and Ψ , the labels u_1 and v_1 coincide and so Δ has a pair of branches $u_1 0^{k_1} \rightarrow u_1 0^{k_2}$.

Let $\alpha = .u_1$. Then the function f fixes α . The components

$$\Psi^{-1} \circ (\Delta_1 + \varepsilon(x^m)) \circ \Psi \quad \text{and} \quad \Psi^{-1} \circ (\varepsilon(x^n) + \Delta_2) \circ \Psi$$

of Δ correspond to the components f_1 and f_2 of f at α , respectively. Indeed, replacing Δ_2 by $\varepsilon(x^m)$ does not affect the pairs of branches of Δ associated with the action of f on dyadic intervals in $[0, \alpha]$. It replaces the pairs of branches of Δ associated with the action of f on the interval $[\alpha, 1]$ by trivial branches; i.e., branches of the form $b \rightarrow b$ for finite binary words b . Clearly, the corresponding function of F is f_1 . A similar argument works for the second component.

In the other direction, assume that a non trivial function $f \in F$ fixes a finite dyadic fraction α . We can assume that f does not fix an open neighborhood of α . Otherwise, one can replace α by a dyadic fraction β such that f fixes the interval $[\alpha, \beta]$ or $[\beta, \alpha]$ and such that f does not fix an open neighborhood of β (note that the components of f at α and

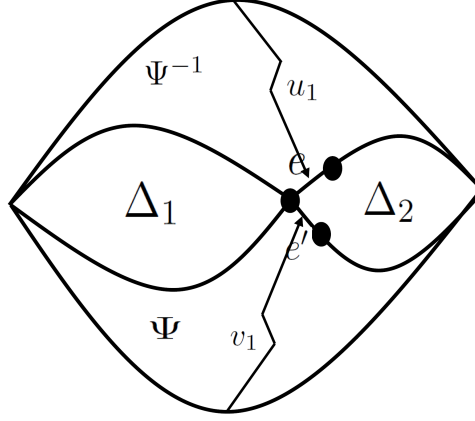


Figure 5.9: The diagram Δ has components as in Definition 3.9

at β coincide in that case). We assume that f does not fix a right neighborhood of α , the argument for f not fixing a left neighborhood of α is similar. Let u be a finite binary word ending with 1 such that $\alpha = .u$. By Lemma 2.6(3), the reduced diagram Δ representing f has a pair of branches $u0^{k_1} \rightarrow u0^{k_2}$ for some $k_1, k_2 \in \mathbb{N}$. In particular, u labels a positive and a negative path on Δ . Let Ψ' be the minimal tree-diagram such that u is a branch of Ψ' and let $\Psi \equiv \Psi'^{-1}$. Then Ψ can be viewed as a subdiagram of Δ^- such that $\mathbf{bot}(\Psi) = \mathbf{bot}(\Delta^-)$ and Ψ^{-1} can be viewed as a subdiagram of Δ^+ such that $\mathbf{top}(\Psi^{-1}) = \mathbf{top}(\Delta^+)$. Let e be the terminal edge of the positive path u in Δ and e' be the terminal edge of the negative path u in Δ . Clearly, e lies on $\mathbf{bot}(\Psi^{-1})$ and e' lies on $\mathbf{top}(\Psi)$. We denote by e_- the initial vertex of e and by e'_- the initial vertex of e' . The vertices e_- and e'_- are vertices on the horizontal 1-path of Δ . The pair of branches $u0^{k_1} \rightarrow u0^{k_2}$ of Δ implies that e_- and e'_- coincide. Thus, if one removes the subdiagrams Ψ^{-1} and Ψ from Δ , the resulting diagram is a sum of two spherical diagrams Δ_1 and Δ_2 such that $\tau(\Delta_1) = \iota(\Delta_2) = e_-$. One can show as above that the components of Δ defined by the subdiagrams $\Psi^{\pm 1}$, Δ_1 and Δ_2 as in Definition 3.9 correspond to the components of f at α . \square

To prove Conjecture 3.11, we prove a stronger result. Namely, that $\text{Cl}(H)$ is generated by the subgroup H and all components of functions in H . We will need the following definition.

Definition 5.3. Let H be a subgroup of F . A function $f \in F$ is said to be *dyadic-piecewise- H* if there exist $n \in \mathbb{N}$, finite dyadic fractions $\alpha_1, \dots, \alpha_{n-1}$ in $(0, 1)$ and functions $h_1, \dots, h_n \in H$ such that

$$f(t) = \begin{cases} h_1(t) & \text{if } t \in [0, \alpha_1] \\ h_2(t) & \text{if } t \in [\alpha_1, \alpha_2] \\ \vdots & \\ h_{n-1}(t) & \text{if } t \in [\alpha_{n-2}, \alpha_{n-1}] \\ h_n(t) & \text{if } t \in [\alpha_{n-1}, 1] \end{cases}$$

In particular, for each $i = 1, \dots, n-1$, we have $h_i(\alpha_i) = h_{i+1}(\alpha_i)$. We say that H is *dyadic-piecewise-closed* if all dyadic-piecewise- H functions belong to H .

Remark 5.4. Let H be a subgroup of F . We let $\text{DPiec}(H)$ be the set of all dyadic-piecewise- H functions. Then $\text{DPiec}(H)$ is a dyadic-piecewise-closed subgroup of F .

Lemma 5.5. Let H be a subgroup of F . Let H_1 be the subgroup of F generated by all elements of H together with all components of functions in H . Let \bar{H} be the minimal subgroup of F which contains H and is closed for components. Then

$$H_1 = \bar{H} = \text{DPiec}(H).$$

Proof. We let $G = \text{DPiec}(H)$. It is obvious that $H_1 \subseteq \bar{H}$. To see that $\bar{H} \subseteq G$ it suffices to note that G is closed for components. Indeed, if $f \in G$ fixes a finite dyadic fraction $\alpha \in (0, 1)$ then the components

$$f_1(t) = \begin{cases} f(t) & \text{if } t \in [0, \alpha] \\ t & \text{if } t \in [\alpha, 1] \end{cases} \quad f_2(t) = \begin{cases} t & \text{if } t \in [0, \alpha] \\ f(t) & \text{if } t \in [\alpha, 1] \end{cases}$$

are dyadic-piecewise- G , as f and the identity belong to G . Since G is dyadic-piecewise-closed, $f_1, f_2 \in G$. Thus, it suffices to prove that $G \subseteq H_1$.

Let $f \in G$ be dyadic-piecewise- H . We prove that f belongs to H_1 by induction on the number n of pieces in f . If $n = 1$, then $f \in H$. If $n = 2$, then there are $h_1, h_2 \in H$ and a finite dyadic fraction $\alpha_1 \in (0, 1)$ such that $h_1(\alpha_1) = h_2(\alpha_1)$ and such that

$$f(t) = \begin{cases} h_1(t) & \text{if } t \in [0, \alpha_1] \\ h_2(t) & \text{if } t \in [\alpha_1, 1] \end{cases}$$

Since $h_1(\alpha_1) = h_2(\alpha_1)$, we have that $h_2 h_1^{-1}(\alpha_1) = \alpha_1$. Since H_1 contains all components of elements in H , the function

$$k(t) = \begin{cases} t & \text{if } t \in [0, \alpha_1] \\ h_2 h_1^{-1}(t) & \text{if } t \in [\alpha_1, 1] \end{cases}$$

belongs to H_1 . It suffices to notice that $f = k h_1 \in H_1$.

For $n > 2$, let $h_1, \dots, h_n \in H$ and $\alpha_1, \dots, \alpha_{n-1} \in (0, 1)$ be finite dyadic fractions such that $h_i(\alpha_i) = h_{i+1}(\alpha_i)$ for all $i = 1, \dots, n-1$ and such that

$$f(t) = \begin{cases} h_1(t) & \text{if } t \in [0, \alpha_1] \\ h_2(t) & \text{if } t \in [\alpha_1, \alpha_2] \\ \vdots & \\ h_{n-1}(t) & \text{if } t \in [\alpha_{n-2}, \alpha_{n-1}] \\ h_n(t) & \text{if } t \in [\alpha_{n-1}, 1] \end{cases}$$

By the induction hypothesis, the function

$$k_1(t) = \begin{cases} h_1(t) & \text{if } t \in [0, \alpha_1] \\ h_2(t) & \text{if } t \in [\alpha_1, \alpha_2] \\ \vdots & \\ h_{n-1}(t) & \text{if } t \in [\alpha_{n-2}, 1] \end{cases}$$

belongs to H_1 .

Since $h_{n-1}(\alpha_{n-1}) = h_n(\alpha_{n-1})$ we have that $h_n h_{n-1}^{-1}(\alpha_{n-1}) = \alpha_{n-1}$. Since H_1 contains all components of functions in H , the function

$$k_2(t) = \begin{cases} t & \text{if } t \in [0, \alpha_{n-1}] \\ h_n h_{n-1}^{-1}(t) & \text{if } t \in [\alpha_{n-1}, 1] \end{cases}$$

belongs to H_1 . It suffices to notice that $f = k_2 k_1$. □

Theorem 5.6. Let H be a subgroup of F . Then $\text{Cl}(H) = \text{DPiec}(H)$. In particular, by Lemma 5.5, the closure of H is the minimal subgroup of F which contains H and is closed for components.

Proof. Since $\text{Cl}(H)$ contains H and is closed for components (see Remark 3.10), by Lemma 5.5, $\text{DPiec}(H) \subseteq \text{Cl}(H)$. To prove the other direction, we show that if $f \in \text{Cl}(H)$, then f is dyadic-piecewise- H .

Let $f \in \text{Cl}(H)$ and let Δ be the reduced diagram of f . Let $u_i \rightarrow v_i$ for $i = 1, \dots, n$ be the pairs of branches of Δ . Since $f \in \text{Cl}(H)$, the diagram Δ is accepted by the core $\mathcal{L}(H)$. It follows that for each $i = 1, \dots, n$, u_i and v_i label paths on $\mathcal{L}(H)$ such that $u_i^+ = v_i^+$. By Lemma 4.6, there exists $k \in \mathbb{N}$, such that for all $i = 1, \dots, n$ and each finite binary word w of length k , there is a function $h_{i,w} \in H$ with a pair of branches $u_i w \rightarrow v_i w$. Notice that all of these pairs of branches are pairs of branches of the function f . Thus, for each $i = 1, \dots, n$ and every finite binary word $w \in \{0, 1\}^k$, f coincides with some function of H on the interval $[u_i w]$. It remains to notice that the dyadic intervals $[u_i w]$ for $i = 1, \dots, n$ and $w \in \{0, 1\}^k$ form a dyadic subdivision of $[0, 1]$. □

We note that since elements of F are piecewise-linear functions where all breakpoints are finite dyadic, Theorem 5.6 can be formulated as Theorem 1.1 from the introduction.

Theorem 5.6 implies the following.

Corollary 5.7. A subgroup H of F is closed if and only if H is closed for components.

Corollary 5.8. Let H be a subgroup of F , then the actions of H and of $\text{Cl}(H)$ on the interval $[0, 1]$ have the same orbits.

6 Transitivity of the action of H on the set \mathcal{D}

Let \mathcal{D} be the set of finite dyadic fractions in $(0, 1)$. In this section we consider the action of a subgroup $H \leq F$ on the set \mathcal{D} . By Corollary 5.8, it is enough to consider the action of $\text{Cl}(H)$ on \mathcal{D} .

We begin with the following lemma.

Lemma 6.1. Let H be a subgroup of F . If u and v label paths on the core $\mathcal{L}(H)$ such that $u^+ = v^+$, then there is a function $f \in \text{Cl}(H)$ with a pair of branches $u \rightarrow v$. Moreover, f is represented by a diagram Δ accepted by $\mathcal{L}(H)$ which has the pair of branches $u \rightarrow v$.

Proof. By Lemma 4.6, there is some $k \in \mathbb{N}$ such that for each finite binary word $w \in \{0, 1\}^k$, there is a function $h_w \in H$ with a pair of branches $uw \rightarrow vw$. Using the functions h_w , one can construct a dyadic-piecewise- H function f , such that for all $w \in \{0, 1\}^k$, f coincides with h_w on the interval $[uw]$. In other words, f takes the branch uw onto vw . Since that is true for all $w \in \{0, 1\}^k$, f has the pair of branches $u \rightarrow v$. By Theorem 5.6, $f \in \text{Cl}(H)$.

For the last statement, let Δ' be a reduced diagram of f . If $u \rightarrow v$ is a pair of branches of Δ' , then we are done, as Δ' is accepted by $\mathcal{L}(H)$. Otherwise, $u \equiv u_1s$ and $v \equiv v_1s$ for some common suffix s , such that $u_1 \rightarrow v_1$ is a pair of branches of Δ' . In particular, on $\mathcal{L}(H)$, $u_1^+ = v_1^+$. Let Δ'' be the minimal diagram of the identity with a pair of branches $s \rightarrow s$. The minimality of Δ'' and the fact that u_1s, v_1s label paths on $\mathcal{L}(H)$ guarantee that for each pair of branches $b \rightarrow b$ of Δ'' , u_1b and v_1b label paths on $\mathcal{L}(H)$. Since $u_1^+ = v_1^+$, we have $(u_1b)^+ = (v_1b)^+$ in $\mathcal{L}(H)$ for each such b . Now, let $u_1^+ = v_1^+ = e$ be the edge on the horizontal 1-path of Δ' . We let Δ be the diagram resulting from Δ' by the replacement of the edge e by the diagram Δ'' . The diagram Δ is equivalent to Δ' and as such represents f . By construction, it has the pair of branches $u \rightarrow v$. The above arguments show that for each pair of branches $w_1 \rightarrow w_2$ of Δ , w_1 and w_2 are paths on $\mathcal{L}(H)$ which terminate on the same edge. Hence, Δ is accepted by $\mathcal{L}(H)$. \square

Lemma 6.2. Let H be a subgroup of F . Let w_1 and w_2 be two finite binary words ending with 1. The finite dyadic fractions $.w_1$ and $.w_2$ belong to the same orbit of the action of H on \mathcal{D} if and only if one of the following (mutually exclusive) conditions holds.

1. The words w_1 and w_2 do not label paths on the core $\mathcal{L}(H)$. In addition, if $w_1 \equiv u_1s_1$ and $w_2 \equiv u_2s_2$ such that u_1 and u_2 are the longest prefixes of w_1 and w_2 which label paths on $\mathcal{L}(H)$, then $u_1^+ = u_2^+$ in $\mathcal{L}(H)$ and the suffixes s_1 and s_2 coincide.
2. There exist $m_1, m_2 \geq 0$ such that the words $w_10^{m_1}$ and $w_20^{m_2}$ label paths on $\mathcal{L}(H)$ and the terminal edges $(w_10^{m_1})^+$ and $(w_20^{m_2})^+$ coincide.

Proof. If condition (1) is satisfied, then by Lemma 6.1, there is a function $f \in \text{Cl}(H)$ with the pair of branches $u_1 \rightarrow u_2$. Since $s_1 \equiv s_2$, f maps the fraction $.w_1 = .u_1s_1$ to $.w_2 = .u_2s_2$. If condition (2) is satisfied, then by Lemma 6.1 there is a function $f \in \text{Cl}(H)$ with a pair of branches $w_10^{m_1} \rightarrow w_20^{m_2}$. In particular, it takes the dyadic fraction $.w_1$ to $.w_2$. Thus, if condition (1) or (2) holds, then $.w_1$ and $.w_2$ belong to the same orbit of the action of $\text{Cl}(H)$ on \mathcal{D} , and thus, by Lemma 5.8, to the same orbit of the action of H on \mathcal{D} .

In the other direction, assume that $.w_1$ and $.w_2$ belong to the same orbit of H and let $h \in H$ be such that $h(.w_1) = .w_2$. Let Δ be a reduced diagram of h . Let u be the unique

positive branch of Δ which is a prefix of $w_1 0^\mathbb{N}$. Let v be the negative branch of Δ such that $u \rightarrow v$ is a pair of branches of Δ . There are two cases to consider.

(1) u is a strict prefix of w_1 . In that case, let $w_1 \equiv us$. Since $h(.us) = .vs = .w_2$ and vs and w_2 both end with the digit 1, we must have $w_2 \equiv vs$. By Lemma 4.8, we have $u^+ = v^+$ on $\mathcal{L}(H)$. Let w be the longest prefix of s such that uw (equiv., vw) labels a path on $\mathcal{L}(H)$. Clearly, $(uw)^+ = (vw)^+$. If $w \equiv s$, then condition (2) in the lemma is satisfied with $m_1 = m_2 = 0$. Otherwise, one can write $s \equiv ws'$. Then condition (1) is satisfied with $u_1 \equiv uw$, $u_2 \equiv vw$ and $s_1 \equiv s_2 \equiv s'$.

(2) $u \equiv w_1 0^{m_1}$ for some $m_1 \geq 0$. In that case, $v \equiv w_2 0^{m_2}$ for some $m_2 \geq 0$. Otherwise, $h(.w_1) = .v \neq .w_2$. By Lemma 4.8, we have $u^+ = v^+$ in $\mathcal{L}(H)$. In other words, condition (2) in the lemma holds. \square

To formulate a simple criterion for the transitivity of the action of a subgroup $H \leq F$ on the set of finite dyadic fractions \mathcal{D} , we make the following definition.

Definition 6.3. Let H be a subgroup of F . We define a directed graph $\Gamma(H)$ as follows. The set of vertices of $\Gamma(H)$ is the set of edges of the core $\mathcal{L}(H)$ which are not incident to $\iota(\mathcal{L}(H))$. For each pair of vertices e, e' in $\Gamma(H)$ there is a directed edge from e to e' in $\Gamma(H)$ if and only if there is a positive cell π in $\mathcal{L}(H)$ such that e is the top edge of π and e' is the left bottom edge of π .

Note that each vertex e of $\Gamma(H)$ can have at most one outgoing edge. Thus we have the following.

Remark 6.4. Let e_1 and e_2 be two vertices of $\Gamma(H)$ which lie in the same connected component of $\Gamma(H)$ (when $\Gamma(H)$ is considered as an unoriented graph). Then there are directed paths p_1 and p_2 in $\Gamma(H)$, with initial vertices e_1 and e_2 respectively, and the same terminal vertex.

Let u and v be two finite binary words which contain the digit 1 and label paths on $\mathcal{L}(H)$. Then u^+ and v^+ are vertices of $\Gamma(H)$ (see Corollary 4.10). By Remark 6.4, u^+ and v^+ belong to the same connected component of $\Gamma(H)$ if and only if, for some $m, n \geq 0$, $u0^m$ and $v0^n$ label paths on $\mathcal{L}(H)$ such that $(u0^m)^+ = (v0^n)^+$.

Theorem 6.5. Let H be a finitely generated subgroup of F . Then the following assertions hold.

- (1) If there is an edge e in $\mathcal{L}(H)$ which is not the top edge of any positive cell in $\mathcal{L}(H)$, then the action of H on \mathcal{D} has infinitely many orbits.
- (2) If every edge e in $\mathcal{L}(H)$ is the top edge of some positive cell in $\mathcal{L}(H)$ then the number of orbits of the action of H on \mathcal{D} is equal to the number of connected components of $\Gamma(H)$ when $\Gamma(H)$ is viewed as an unoriented graph.

Proof. To prove part (1), assume that there is an edge e in $\mathcal{L}(H)$ which is not the top edge of any positive cell in the core. Let u be a path on $\mathcal{L}(H)$ such that $u^+ = e$. Clearly, if u is a strict prefix of a finite binary word v then v does not label a path on $\mathcal{L}(H)$. We consider the infinite set of finite dyadic fractions $B = \{.u1^n : n \in \mathbb{N}\}$. Lemma 6.2 implies that any two distinct fractions in B do not belong to the same orbit of H . Thus, the action of H on \mathcal{D} has infinitely many orbits as required.

For part (2), assume that every edge e in $\mathcal{L}(H)$ is the top edge of some positive cell π . Then every finite binary word u labels a path on $\mathcal{L}(H)$.

Let α_1 and α_2 be finite dyadic fractions. Then α_1 and α_2 can be written uniquely in binary form as $.w_1$ and $.w_2$ where w_1 and w_2 have suffix 1. In particular, $e_1 = w_1^+$ and $e_2 = w_2^+$ are edges of $\mathcal{L}(H)$ which are not incident to $\iota(\mathcal{L}(H))$. We claim that α_1 and α_2 belong to the same orbit of the action of H on \mathcal{D} if and only if e_1 and e_2 belong to the same connected component of $\Gamma(H)$. That would complete the proof of the theorem. (Indeed, every connected component of $\Gamma(H)$ contains an edge w^+ of \mathcal{L} for some word w ending with 1.)

Since w_1 and w_2 label paths in $\mathcal{L}(H)$, by Lemma 6.2, $.w_1$ and $.w_2$ belong to the same orbit of H if and only if for some $m_1, m_2 \geq 0$, we have $(w_1 0^{m_1})^+ = (w_2 0^{m_2})^+$ in $\mathcal{L}(H)$. Notice that for all $m_1, m_2 \geq 0$, the vertex $(w_1 0^{m_1})^+$ (resp. $(w_2 0^{m_2})^+$) of $\Gamma(H)$ belongs to the same connected component as the vertex w_1^+ (resp. w_2^+). Thus, if there exist $m_1, m_2 \geq 0$ as described, we get that w_1^+ and w_2^+ both belong to the connected component in $\Gamma(H)$ of $(w_1 0^{m_1})^+$.

In the other direction, assume that w_1^+ and w_2^+ belong to the same connected component of $\Gamma(H)$. Then by Remark 6.4 and the comments succeeding it, for some $m, n \geq 0$, the edges $(w_1 0^m)^+$ and $(w_2 0^n)^+$ coincide in $\mathcal{L}(H)$. By Lemma 6.1, there is an element $k \in \text{Cl}(H)$ with a pair of branches $w_1 0^m \rightarrow w_2 0^n$. In particular, $\alpha_1 = .w_1$ and $\alpha_2 = .w_2$ belong to the same orbit of the action of $\text{Cl}(H)$, and thus of H , on the set of finite dyadic fractions \mathcal{D} . \square

If H is a finitely generated subgroup of F then Theorem 6.5 gives an algorithm for deciding the transitivity of the action of H on \mathcal{D} . We note that an edge e of $\mathcal{L}(H)$ is a vertex of $\Gamma(H)$ if and only if it is an outgoing edge of some inner vertex in $\mathcal{L}(H)$. Condition (2) in Theorem 6.5 can be formulated in terms of the number of inner vertices of the core $\mathcal{L}(H)$ using the following proposition.

The definition of inner edges and inner vertices of $\mathcal{L}(H)$ extends naturally to all 2-automata considered in the proof of the following proposition. If e is an edge in a directed 2-complex \mathcal{L} , we denote by e_- the initial vertex $\iota(e)$ and by e_+ the terminal vertex $\tau(e)$.

Proposition 6.6. Let H be a subgroup of F and let e_1, e_2 be edges of $\mathcal{L}(H)$ which are not incident to $\iota(\mathcal{L}(H))$. Then e_1, e_2 belong to the same connected component of $\Gamma(H)$ (when viewed as an unoriented graph), if and only if $e_{1-} = e_{2-}$ in $\mathcal{L}(H)$.

Proof. To prove that if e_1 and e_2 belong to the same connected component of $\Gamma(H)$ then $e_{1-} = e_{2-}$ it suffices to consider the case where e_1 and e_2 are adjacent vertices of $\Gamma(H)$. Without loss of generality, we can assume that there is a positive cell π in $\mathcal{L}(H)$ with top edge e_1 and bottom edge e_2 . Clearly, that implies that $e_{1-} = e_{2-} = \iota(\pi)$ in $\mathcal{L}(H)$. To prove the other direction, we consider the construction of the core $\mathcal{L}(H)$. Let \mathcal{L}'_n , $n \geq 0$ be the 2-automata constructed in the process of constructing $\mathcal{L}(H)$, as defined in the proof of Lemma 4.6. It suffices to prove the following lemma.

Lemma 6.7. For all $n \geq 0$, if e_1 and e_2 are edges of \mathcal{L}'_n such that $e_{1-} = e_{2-} \neq \iota(\mathcal{L}'_n)$, then the images of e_1 and e_2 in $\mathcal{L}(H)$ (under the natural morphism) belong to the same connected component of $\Gamma(H)$.

Proof. To simplify notation we denote the morphism from \mathcal{L}'_n to $\mathcal{L}(H)$ by ψ , with no reference to the index n . This should not cause confusion as the morphisms from \mathcal{L}'_n to $\mathcal{L}(H)$ are compatible with the natural morphisms from \mathcal{L}'_i to \mathcal{L}'_j for $i < j$.

We prove the lemma by induction on n . If $n = 0$ then \mathcal{L}'_0 is a bouquet of the generating diagrams Δ_i (each, with the top and bottom edges identified). Let e_1 and e_2 be edges of \mathcal{L}'_0 such that $e_{1-} = e_{2-} \neq \iota(\mathcal{L}'_0)$. Then e_1 and e_2 are edges in one of the diagrams Δ_i . Let e' be the outgoing edge of $e_{1-} = e_{2-}$ which lies on the horizontal 1-path of Δ_i . It is easy to see that $\psi(e_1)$ and $\psi(e')$ belong to the same connected component of $\Gamma(H)$. Similarly, for $\psi(e_2)$ and $\psi(e')$. Thus, $\psi(e_1)$ and $\psi(e_2)$ belong to the same connected component of $\Gamma(H)$.

Now, let $n \in \mathbb{N}$ and assume that the claim holds for $n - 1$. We first consider the case where the n^{th} folding is a folding of type 2. In that case, no vertices are identified in the transition from \mathcal{L}'_{n-1} to \mathcal{L}'_n . Let e_1 and e_2 be edges in \mathcal{L}'_n such that $e_{1-} = e_{2-} \neq \iota(\mathcal{L}'_n)$. Then e_1 and e_2 can be lifted to edges e'_1 and e'_2 in \mathcal{L}'_{n-1} . Clearly, $e'_{1-} = e'_{2-} \neq \iota(\mathcal{L}'_{n-1})$. Thus, by induction we have $\psi(e_1) = \psi(e'_1) = \psi(e'_2) = \psi(e_2)$, as required.

Therefore, we can assume that the n^{th} folding is of type 1. Let π_1 and π_2 be the positive cells of \mathcal{L}'_{n-1} which are folded in the transition to \mathcal{L}'_n . Let x_1, x_2, x_3 be the vertices on $\mathbf{bot}(\pi_1)$ from left to right. Similarly, let y_1, y_2, y_3 be the vertices on $\mathbf{bot}(\pi_2)$ from left to right. Then in the transition to \mathcal{L}'_n , for $i = 1, 2, 3$, x_i is identified with y_i , to give a vertex z_i of \mathcal{L}'_n .

Let e_1 and e_2 be edges of \mathcal{L}'_n such that $e_{1-} = e_{2-} \neq \iota(\mathcal{L}'_n)$. We can assume that $e_{1-} = e_{2-} = z_i$, for some $i \in \{1, 2, 3\}$. Otherwise, we are done by the induction hypothesis, as in the case of a folding of type 2. Assume first that $e_{1-} = e_{2-} = z_1$. The edges e_1 and e_2 can be lifted to edges e'_1, e'_2 in \mathcal{L}'_{n-1} . Clearly, $e'_{1-}, e'_{2-} \in \{x_1, y_1\}$. If $e'_{1-} = e'_{2-}$, we are done by the induction hypothesis. Thus, assume that $e'_{1-} = x_1$ and $e'_{2-} = y_1$. Since $\mathbf{top}(\pi_1)_{-} = x_1$ and $\mathbf{top}(\pi_2)_{-} = y_1$, we have by induction, that $\psi(e'_1)$ and $\psi(\mathbf{top}(\pi_1))$ belong to the same connected component of $\Gamma(H)$. Similarly, $\psi(e'_2)$ and $\psi(\mathbf{top}(\pi_2))$ belong to the same connected component of $\Gamma(H)$. Since $\psi(\mathbf{top}(\pi_1)) = \psi(\mathbf{top}(\pi_2))$, we get that $\psi(e_1) = \psi(e'_1)$ belongs to the same connected component of $\psi(e_2) = \psi(e'_2)$, as required.

If $e_{1-} = e_{2-} = z_2$, then the same argument works if one replaces the top edges of π_1 and π_2 by the right bottom edges of π_1 and π_2 (which have initial vertices x_2 and y_2).

Assume that $e_{1-} = e_{2-} = z_3$. In particular, we assume that z_3 is an inner vertex of \mathcal{L}'_n . To adapt the argument used for $i = 1, 2$ for this case it suffices to show that there are edges p_1, p_2 in \mathcal{L}'_{n-1} such that $p_{1-} = x_3$, $p_{2-} = y_3$ and such that $\psi(p_1)$ and $\psi(p_2)$ belong to the same connected component of $\Gamma(H)$. In practice, we would consider edges p'_1 and p'_2 of \mathcal{L}'_0 which project onto edges p_1, p_2 as described.

Let π'_1 and π'_2 be liftings of the cells π_1 and π_2 to the bouquet of spheres \mathcal{L}'_0 . We consider the cell π'_1 . It is a cell of one of the diagrams Δ_i . To simplify notation, we assume that π'_1 is a cell in Δ_i^+ . The vertex $\tau(\pi'_1)$ projects onto the vertex x_3 in \mathcal{L}'_n . We let p'_1 be the top-most outgoing edge of $\tau(\pi'_1)$ in the diagram Δ_i . Clearly, p'_1 is mapped to an outgoing edge of x_3 by the morphism from \mathcal{L}'_0 to \mathcal{L}'_{n-1} . In a similar way, we can assume that π'_2 is a cell in a positive subdiagram Δ_j^+ in the bouquet of spheres \mathcal{L}'_0 . We let p'_2 be the top-most outgoing edge of $\tau(\pi'_2)$ in Δ_j . It suffices to prove that $\psi(p'_1)$ and $\psi(p'_2)$ belong to the same connected component of $\Gamma(H)$.

Consider the edge p'_1 . Since p'_1 is the top-most outgoing edge of $\tau(\pi'_1)$, it is the right bottom edge of some cell π in Δ_i^+ . Let a be the left bottom edge of π . Then a lies above

$\mathbf{top}(\pi'_1)$ (or coincides with it) in Δ_i and $a_+ = \mathbf{top}(\pi'_1)_+$. Let u be the positive path in Δ_i with terminal edge $\mathbf{top}(\pi'_1)$. Then u can be written in the form $u \equiv u_1 0 1^k$ for some word u_1 and $k \geq 0$ (indeed, $\mathbf{top}(\pi'_1)$ is not incident to $\tau(\Delta_i)$ and as such u contains the digit 0). We note that the positive path $u_1 0$ in Δ_i terminates on the edge a , i.e., on the left bottom edge of π . Thus, $u_1 1$ is a positive path in Δ_i with terminal edge p'_1 . Similarly, there is a finite binary word v_1 , such that the positive path to $\mathbf{top}(\pi'_2)$ in Δ_j is $v_1 0 1^r$ for some $r \geq 0$ and the positive path $v_1 1$ on Δ_j terminates on p'_2 .

Since $\psi(\mathbf{top}(\pi'_1)) = \psi(\mathbf{top}(\pi'_2))$ in $\mathcal{L}(H)$, the paths $u_1 0 1^k$ and $v_1 0 1^r$ on $\mathcal{L}(H)$ terminate on the same edge. Thus, by Lemma 6.1, there is a diagram Δ accepted by $\mathcal{L}(H)$ with a pair of branches $u_1 0 1^k \rightarrow v_1 0 1^r$. Then the following pair of branches of Δ is of the form $u_1 1 0^{k_1} \rightarrow v_1 1 0^{r_1}$ for some $k_1, r_1 \geq 0$. By Lemma 4.8, $(u_1 1 0^{k_1})^+ = (v_1 1 0^{r_1})^+$ on $\mathcal{L}(H)$. We denote this edge of $\mathcal{L}(H)$ by the letter b . In $\mathcal{L}(H)$ we have $\psi(p'_1) = (u_1 1)^+$ and $\psi(p'_2) = (v_1 1)^+$. Since $(u_1 1)^+$ and $(v_1 1)^+$ belong to the same connected component of b in $\Gamma(H)$, $\psi(p'_1)$ and $\psi(p'_2)$ belong to the same connected component of $\Gamma(H)$, as required. \square

\square

Since any inner vertex in $\mathcal{L}(H)$ has at least one outgoing edge, the following is an immediate corollary of Theorem 6.5 and Proposition 6.6.

Corollary 6.8. Let H be a finitely generated subgroup of F . Then the following assertions hold.

- (1) If there is an edge e in $\mathcal{L}(H)$ which is not the top edge of any positive cell in $\mathcal{L}(H)$, then the action of H on \mathcal{D} has infinitely many orbits.
- (2) If every edge e in $\mathcal{L}(H)$ is the top edge of some positive cell in $\mathcal{L}(H)$ then the number of orbits of the action of H on \mathcal{D} is equal to the number of inner vertices of $\mathcal{L}(H)$.

In particular, we have the following.

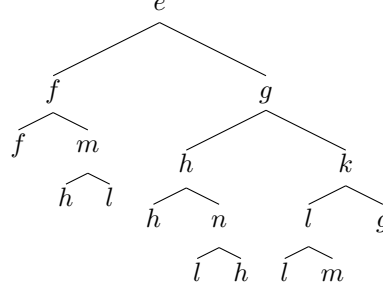
Corollary 6.9. Let $H \leq F$. Then H acts transitively on \mathcal{D} if and only if the following assertions hold.

- (1) Every edge in $\mathcal{L}(H)$ is the top edge of some positive cell in the core.
- (2) There is a unique inner vertex in $\mathcal{L}(H)$.

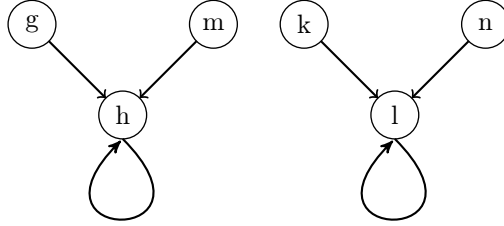
In practice, constructing the graph $\Gamma(H)$ is a simple way to count the number of inner vertices of $\mathcal{L}(H)$. Thus, we would often apply Theorem 6.5.

Example 6.10. The action of the subgroup $H = \langle x_0 x_1, x_1 x_2, x_2 x_3 \rangle$ of F on the set of finite dyadic fractions \mathcal{D} has two orbits.

Proof. The core $\mathcal{L}(H)$ can be described by the following binary tree.



We note that every edge of $\mathcal{L}(H)$ is the top edge of some positive cell. The graph $\Gamma(H)$ is the following.



Thus, by Theorem 6.5, the action of H on \mathcal{D} has two orbits. □

Remark 6.11. The subgroup H from Example 6.10 is the group \vec{F} recently defined by Jones, in his study of the relation between Thompson group F and links [20]. It was proved in [20] (see also [14]) that the action of \vec{F} on \mathcal{D} has two orbits. Two finite dyadic fractions α_1 and α_2 belong to the same orbit if and only if the sum of digits in their binary representation is equal modulo 2. Considering the core $\mathcal{L}(\vec{F})$ and the proof of Theorem 6.5 it is easy to see that this is the case.

7 The generation problem in F

Let X be a finite subset of F . We are interested in determining whether X generates F . Let $H = \langle X \rangle$. We make the observation that $H = F$ if and only if (1) $H[F, F] = F$ and (2) $[F, F] \subseteq H$. To determine if $H[F, F] = F$ it suffices to consider the image of H in the abelianization of F . Thus, the generation problem in F reduces to determining whether a finitely generated subgroup H contains the derived subgroup of F . (Since $[F, F]$ is simple and the center of F is trivial [11], this is equivalent to determining if H is a normal subgroup of F .) We start with a condition for $\text{Cl}(H)$ to contain $[F, F]$.

Lemma 7.1. Let $H \leq F$ be a subgroup of F . Then $\text{Cl}(H)$ contains the derived subgroup of F if and only the following assertions hold.

- (1) Every finite binary word u labels a path on $\mathcal{L}(H)$.
- (2) For any pair of finite binary words u and v which contain both digits 0 and 1, we have $u^+ = v^+$ on $\mathcal{L}(H)$.

Equivalently, in terms of the structure of the core, $\text{Cl}(H)$ is a normal subgroup of F if and only if

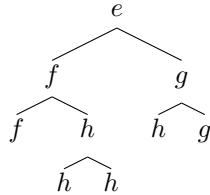
- (1') every edge in $\mathcal{L}(H)$ is the top edge of some positive cell; and
- (2') there is a unique inner edge in $\text{Cl}(H)$.

Proof. Condition (1') is equivalent to the condition that every finite binary word u labels a path on $\mathcal{L}(H)$. When condition (1') holds, condition (2') is equivalent to condition (2) by Corollary 4.10.

Assume that $\text{Cl}(H)$ contains the derived subgroup of F . Then every reduced diagram in $[F, F]$ is accepted by $\mathcal{L}(H)$. Recall that $[F, F]$ is the subgroup of F of all functions with slope 1 both at 0^+ and at 1^- . For any pair of finite binary words u and v which contain both digits 0 and 1, there is an element $f \in [F, F]$ which maps $[u]$ linearly onto $[v]$. It is easy to construct such an element so that the reduced diagram of f has the pair of branches $u \rightarrow v$. It follows from Lemma 4.8 that u and v label paths in $\mathcal{L}(H)$ such that $u^+ = v^+$. Thus, condition (2) is satisfied. For (1) it suffices to show that for all $n \in \mathbb{N}$, 0^n and 1^n also label paths on $\mathcal{L}(H)$. Since $0^n 1$ and $1^n 0$ are paths on the core, we have the result.

Conversely, if (1) and (2) are satisfied, then every diagram in $[F, F]$ is accepted by the core $\mathcal{L}(H)$. Indeed, a diagram Δ is accepted by $\mathcal{L}(H)$ if and only if for any pair of branches $u \rightarrow v$ in Δ , both u and v label paths on $\mathcal{L}(H)$ and $u^+ = v^+$ in $\mathcal{L}(H)$. If $\Delta \in [F, F]$, then the first and last branches of Δ are of the form $0^m \rightarrow 0^m$ and $1^n \rightarrow 1^n$ for some $m, n \in \mathbb{N}$. All other pairs of branches $u_i \rightarrow v_i$ are such that both u_i and v_i contain the digits 0 and 1. Conditions (1) and (2) clearly imply that Δ is accepted by $\mathcal{L}(H)$ and thus that $[F, F] \subseteq \text{Cl}(H)$. \square

Remark 7.2. The core $\mathcal{L}(F)$ of Thompson group F is given by the finite binary tree



In particular, it has a unique inner edge, a distinguished edge, a left boundary edge and a right boundary edge. It is the smallest core (in terms of the number of edges and cells) of a non-trivial subgroup $H \leq F$.

Recall that by Lemma 4.6, if H is a subgroup of F and u, v label paths on $\mathcal{L}(H)$ such that $u^+ = v^+$, then for any long enough extension w , $uw \rightarrow vw$ is a pair of branches of some element in H . We define 2 additional 2-automata related to a subgroup H of F , where the goal is to have this property for paths on the core, with no need to consider extensions. We will need the following remark.

Remark 7.3. Let \mathcal{L}' be a 2-automaton over the Duncie hat \mathcal{K} to which no foldings of type 1 are applicable. If \mathcal{L} results from \mathcal{L}' by a folding of type 2 then no folding of type 1 is applicable to \mathcal{L} .

Definition 7.4. Let H be a subgroup of F generated by a set $X = \{\Delta_i : i \in \mathcal{I}\}$ of reduced diagrams. We define a 2-automaton over \mathcal{K} as follows. First, we identify all $\text{top}(\Delta_i)$ with all $\text{bot}(\Delta_i)$ and obtain a 2-automaton \mathcal{L}' over \mathcal{K} with the distinguished 1-paths $p_{\mathcal{L}'} = q_{\mathcal{L}'}$

$\text{top}(\Delta_i) = \text{bot}(\Delta_i)$. Then we apply foldings of type 1 to \mathcal{L}' as long as possible. The resulting 2-automaton $\mathcal{L}_{\text{sem}}(X)$, to which no folding of type 1 is applicable, is called the *semi-core* of H associated with the generating set X .

The semi-core of H associated with a generating set X depends only on X ; i.e., it does not depend on the order in which foldings were applied to \mathcal{L}' . Remark 7.3 implies that if one applies all possible foldings of type 2 to $\mathcal{L}_{\text{sem}}(X)$, the resulting 2-automaton is the core $\mathcal{L}(H)$. In particular, there is a natural surjective morphism ψ from $\mathcal{L}_{\text{sem}}(X)$ to $\mathcal{L}(H)$. By Lemma 4.4, ψ gives a 1 – 1 correspondence between paths on $\mathcal{L}_{\text{sem}}(X)$ and paths on $\mathcal{L}(H)$. It follows that a finite binary word u labels a path on $\mathcal{L}_{\text{sem}}(X)$ if and only if it labels a path on the core $\mathcal{L}(H)$ and that each word u labels at most one path on the semi-core.

The proof of Lemma 4.6 implies the following.

Lemma 7.5. Let H be a subgroup of F generated by a set of reduced diagrams X . Let u and v be finite binary words which label paths in the semi-core $\mathcal{L}_{\text{sem}}(X)$ such that $u^+ = v^+$. Then there is a function $h \in H$ with a pair of branches $u \rightarrow v$.

Indeed, no foldings of type 2 were applied in the construction of $\mathcal{L}_{\text{sem}}(X)$.

Let H be a subgroup of F generated by a set of reduced diagrams X . Let $\mathcal{L}_{\text{sem}}(X)$ be the associated semi-core of H . We define a 2-automaton $\mathcal{L}_{\text{bra}}(H)$ as follows. For each edge e in $\mathcal{L}_{\text{sem}}(X)$, let u_e be a path on $\mathcal{L}_{\text{sem}}(X)$ such that $u_e^+ = e$ (such a path clearly exists). One can define an equivalence relation R on the set of edges of $\mathcal{L}_{\text{sem}}(X)$, where two edges e_1, e_2 are equivalent if and only if there is an element $h \in H$ with a pair of branches $u_{e_1} \rightarrow u_{e_2}$. Lemma 7.5 implies that R does not depend on the choice of paths u_e . Let ψ be the morphism from $\mathcal{L}_{\text{sem}}(X)$ to $\mathcal{L}(H)$. Lemma 4.8 implies that if e_1 and e_2 are equivalent modulo R , then $\psi(e_1) = \psi(e_2)$. Thus, we can identify all equivalent edges of $\mathcal{L}_{\text{sem}}(X)$ and ψ would induce a morphism from the resulting 2-automaton to $\mathcal{L}(H)$. Let \mathcal{L} be the resulting 2-automaton. We identify cells of \mathcal{L} which share their top path as well as their bottom path and call the resulting 2-automaton the *branches-core* of H , denoted by $\mathcal{L}_{\text{bra}}(H)$. Since $\mathcal{L}(H)$ does not contain two distinct cells with the same top and bottom paths, it is obvious that ψ induces a morphism from $\mathcal{L}_{\text{bra}}(H)$ to $\mathcal{L}(H)$.

The construction of $\mathcal{L}_{\text{bra}}(H)$ from $\mathcal{L}_{\text{sem}}(X)$ implies that if one applies all possible foldings of type 2 to $\mathcal{L}_{\text{bra}}(H)$, the result is the core $\mathcal{L}(H)$. Indeed, this is already true for $\mathcal{L}_{\text{sem}}(X)$ and in the construction of $\mathcal{L}_{\text{bra}}(H)$ from $\mathcal{L}_{\text{sem}}(X)$ only edges (resp. cells) which become identified in $\mathcal{L}(H)$ can become identified in $\mathcal{L}_{\text{bra}}(H)$. Thus, by Lemma 4.4, there is a 1 – 1 correspondence between paths on $\mathcal{L}_{\text{bra}}(H)$ and paths on $\mathcal{L}(H)$. In particular, a finite binary word u labels a path on $\mathcal{L}_{\text{bra}}(H)$ if and only if it labels a path on $\mathcal{L}(H)$ and every finite binary word u labels at most one path on $\mathcal{L}_{\text{bra}}(H)$. The following is immediate from the construction and is the reason for the name of the branches-core.

Lemma 7.6. Let H be a subgroup of F and let u and v be paths on $\mathcal{L}_{\text{bra}}(H)$. Then $u^+ = v^+$ if and only if there is an element $h \in H$ with a pair of branches $u \rightarrow v$.

Proof. As noted above, there is a 1 – 1 correspondence between paths on $\mathcal{L}_{\text{sem}}(X)$ to paths on $\mathcal{L}(H)$ to paths on $\mathcal{L}_{\text{bra}}(H)$. Let u and v be paths on $\mathcal{L}_{\text{bra}}(H)$ such that $u^+ = v^+$ in $\mathcal{L}_{\text{bra}}(H)$. We consider u and v as paths on $\mathcal{L}_{\text{sem}}(X)$. If $u^+ = v^+$, then by Lemma 7.5, there is an element $h \in H$ with a pair of branches $u \rightarrow v$. Otherwise, let $e_1 = u^+$ and $e_2 = v^+$ in

$\mathcal{L}_{\text{sem}}(X)$. Let u_{e_1} and u_{e_2} be the paths on $\mathcal{L}_{\text{sem}}(X)$ chosen in the construction of $\mathcal{L}_{\text{bra}}(H)$, so that $u_{e_1}^+ = e_1 = u^+$ and $u_{e_2}^+ = e_2 = v^+$. The edges e_1 and e_2 being identified in $\mathcal{L}_{\text{bra}}(H)$, means that there is an element $h_2 \in H$ with a pair of branches $u_{e_1} \rightarrow u_{e_2}$. By Lemma 7.5, there is an element $h_1 \in H$ with a pair of branches $u \rightarrow u_{e_1}$ and an element $h_3 \in H$ with a pair of branches $u_{e_2} \rightarrow v$. Hence $h_1 h_2 h_3 \in H$ has the pair of branches $u \rightarrow v$, as required. The proof in the other direction is similar. \square

Lemma 7.6 and the 1 – 1 correspondence between paths on $\mathcal{L}(H)$ and paths on $\mathcal{L}_{\text{bra}}(H)$ imply that the branches-core of H is determined uniquely by H .

We note that if H is generated by a finite set of reduced diagrams X , then one can construct the core $\mathcal{L}(H)$ and the semi-core of H associated with X . We do not know how to construct the branches-core of H , but studying it with relation to the core $\mathcal{L}(H)$ is useful.

Lemma 7.7. Let H be a subgroup of F such that $\text{Cl}(H)$ contains the derived subgroup $[F, F]$. Let $\mathcal{L}_{\text{bra}}(H)$ be the branches-core of H . Then there is a positive cell π in $\mathcal{L}_{\text{bra}}(H)$ such that the top edge of π coincides with the 2 bottom edges of π .

Proof. Since $\text{Cl}(H)$ contains $[F, F]$, by Lemma 7.1, every finite binary word u labels a path on $\mathcal{L}(H)$ (and thus, also on $\mathcal{L}_{\text{bra}}(H)$). Thus, as noted above, for any word u there is a unique path on $\mathcal{L}_{\text{bra}}(H)$ labeled by u . By Lemma 4.4, The same is true for any 2-automaton \mathcal{L}' resulting from $\mathcal{L}_{\text{bra}}(H)$ by applications of foldings of type 2.

Consider the words 01 and 010. By Lemmas 4.6 and 7.1, there is $k \in \mathbb{N}$ for which there is an element in H with a pair of branches $010^k \rightarrow 0100^k \equiv 010^{k+1}$. It follows that the edges $e = (010^k)^+$ and $e_1 = (010^{k+1})^+$ of $\mathcal{L}_{\text{bra}}(H)$ coincide. Clearly, the edge e is the top edge of some positive cell π in $\mathcal{L}_{\text{bra}}(H)$. e_1 is the left bottom edge of π . It suffices to prove that the right bottom edge $e_2 = (010^k 1)^+$ coincides with $e = e_1$.

Assume by contradiction that $e_2 \neq e$. Since the images of e_2 and e in $\mathcal{L}(H)$ coincide, there is a finite sequence S of foldings of type 2 such that, when applied to $\mathcal{L}_{\text{bra}}(H)$, S results in the identification of e_2 and e . We take S to be such a minimal sequence of foldings and consider the state of the automaton $\mathcal{L}_{\text{bra}}(H)$ right before the last folding in S is applied to it. We denote this automaton by \mathcal{L}' and observe that in \mathcal{L}' , $e_2 \neq e$ (when we refer to edges and cells of $\mathcal{L}_{\text{bra}}(H)$ as edges and cells of \mathcal{L}' , the meaning should be clear).

Since a folding of type 2 can be applied to \mathcal{L}' to identify e_2 and e , it follows that in \mathcal{L}' , e_2 is the top edge of a positive cell π' such that $\text{bot}(\pi') = \text{bot}(\pi)$. Since the bottom right edge of π is e_2 , the same is true for π' . In other words, the top edge of π' coincides with its right bottom edge.

Let $w \equiv 010^k$. We claim that for every path $u \equiv wv$ for some finite binary word v , the terminal edge u^+ in \mathcal{L}' is either e or e_2 . If the last digit of u is 0 then $u^+ = e$, otherwise, $u^+ = e_2$. Indeed, this is clearly true if v is empty. If v is of length n for $n \geq 1$ then $v \equiv v'a$ where a is the last digit of v . By induction, $(wv')^+$ is either e or e_2 . Thus, if $a \equiv 0$ then u^+ is the left bottom edge of either π or π' . In either case, $u^+ = e$. Similarly, if $a \equiv 1$ we get that $u^+ = e_2$.

Note that in $\mathcal{L}(H)$, we have $w^+ = (w1)^+$. Thus, by Lemma 4.6, for some $k' \geq 0$ there is an element $h \in H$ with a pair of branches $w1^{k'} \rightarrow w1^{k'+1}$. In particular, h fixes the finite dyadic fraction $\alpha = .010^{k-1}1$ (indeed, $w \equiv 010^k$) and $h'(\alpha^-) = \frac{1}{2}$.

Let Δ be a reduced diagram of h . By Lemma 2.6(4), the diagram Δ has a pair of branches $w1^n \rightarrow w1^{n+1}$ for some $n \geq 0$. (In the notations of 2.6(4), $u \equiv 010^{k-1}1$, $u' \equiv 010^{k-1}$ and $\ell = -1$. Thus, Δ has a pair of branches of the form $u'01^n \equiv w1^n \rightarrow u'01^{n+1} \equiv w1^{n+1}$). We claim that $n > 0$. Indeed, if $n = 0$ then on $\mathcal{L}_{\text{bra}}(H)$, and thus on \mathcal{L}' , we have $w^+ = (w1)^+$, in contradiction to the fact that on \mathcal{L}' , $w^+ = e$ and $(w1)^+ = e_2$. Thus, $n > 0$.

Let u_1, \dots, u_r (resp. v_1, \dots, v_m) be the positive (resp. negative) branches of Δ , with prefix w , ordered from left to right. Since $n > 0$, we have $r, m > 1$. Clearly, $u_r \equiv w1^n$ and $v_m \equiv w1^{n+1}$ so that $u_r \rightarrow v_m$ is a pair of branches of Δ . We assume that $r \leq m$ (otherwise, one can consider Δ^{-1}). There is a pair of consecutive positive branches u_i, u_{i+1} such that the edges u_i^+, u_{i+1}^+ on the bottom path of Δ^+ form the bottom path of an (x, x^2) -cell π_1 in Δ^+ . Indeed, if one deletes the prefix w from each of the branches u_1, \dots, u_r , the result is the set of branches of a subdiagram of Δ^+ .

Let $j = m - (r - i)$. Then v_j, v_{j+1} are the negative branches of Δ such that $u_i \rightarrow v_j$ and $u_{i+1} \rightarrow v_{j+1}$ are pairs of branches of Δ . We claim that the edges v_j^+ and v_{j+1}^+ on $\text{top}(\Delta^-)$ (which coincide with u_i^+ and u_{i+1}^+ as edges of Δ), form the top path of an (x^2, x) -cell of Δ^- . That will give a contradiction to the assumption that Δ is reduced.

Since u_i^+ and u_{i+1}^+ form the bottom path of a cell in Δ^+ , the last letter in u_i is 0 and the last letter in u_{i+1} is 1. Let p_1 and p_2 be the paths on \mathcal{L}' labeled by u_i and u_{i+1} respectively. Then $p_1^+ = e$ and $p_2^+ = e_2$ (indeed, w is a prefix of u_i and u_{i+1}). If q_1 and q_2 are the paths on \mathcal{L}' labeled by v_j and v_{j+1} respectively, then we must have $p_1^+ = q_1^+$ and $p_2^+ = q_2^+$ on \mathcal{L}' . Otherwise, the corresponding paths on $\mathcal{L}_{\text{bra}}(H)$ would not have the same terminal edge, in contradiction to $h \in H$ having the pairs of branches $u_i \rightarrow v_j$ and $u_{i+1} \rightarrow v_{j+1}$.

It follows that $q_1^+ = e$ and $q_2^+ = e_2$ which in turn implies that the label v_j ends with 0 and that v_{j+1} ends with 1. Two consecutive negative branches v_j, v_{j+1} of a diagram satisfy this property if and only if the terminal edges v_j^+ and v_{j+1}^+ form the top path of an (x^2, x) -cell in the diagram. Thus, Δ is not reduced and the proof of the lemma is complete. \square

Corollary 7.8. Let $H \leq F$ be a subgroup such that $\text{Cl}(H)$ contains the derived subgroup of F . Let u and v be finite binary words which contain both digits 0 and 1. Then there exists an integer $k \geq 0$ such that for each pair of finite binary words w_1, w_2 of length $\geq k$ there is an element $h \in H$ with a pair of branches $uw_1 \rightarrow vw_2$.

Proof. We consider the branches-core $\mathcal{L}_{\text{bra}}(H)$. By Lemma 7.7 there is a positive cell π in $\mathcal{L}_{\text{bra}}(H)$ such that the top edge and bottom edges of π coincide. We denote this edge by e . It is clear that e is an inner edge of $\mathcal{L}_{\text{bra}}(H)$. (Inner edges of $\mathcal{L}_{\text{bra}}(H)$ are defined in a similar way to inner edges of $\mathcal{L}(H)$.) Let w be a path on $\mathcal{L}_{\text{bra}}(H)$ such that $w^+ = e$. It follows that for any binary word w' , the path ww' on $\mathcal{L}_{\text{bra}}(H)$ terminates on the edge e . Thus, by Lemma 7.6, for any pair of binary words w_1, w_2 there is an element h_{ww_1, ww_2} in H with a pair of branches $ww_1 \rightarrow ww_2$.

Now let u and v be finite binary words which contain both digits 0 and 1. Let p_u, p_v and p_w be the paths on $\mathcal{L}(H)$ with labels u, v and w respectively. Since u, v and w all contain both digits 0 and 1, by Lemma 7.1, $p_u^+ = p_v^+ = p_w^+$ in $\mathcal{L}(H)$. Then by Lemma 4.6 there exists $k \geq 0$ such that for any finite binary word s of length $\geq k$ there are elements $h_{us, ws}$ and $h_{vs, ws}$ in H with pairs of branches $us \rightarrow ws$ and $vs \rightarrow ws$, respectively.

We claim that the lemma holds for k . Indeed, let w_1, w_2 be a pair of binary words of length $\geq k$. Then the element $h_{uw_1, ww_1} h_{ww_1, ww_2} h_{vw_2, ww_2}^{-1}$ is an element of H with a pair of branches $uw_1 \rightarrow vw_2$. \square

Let H be a subgroup of F and let J be a closed sub-interval of $[0, 1]$. We denote by H_J the subgroup of H of all functions which fix the interval J pointwise. Recall that a subgroup $G \leq F$ has an orbital (a, b) if it fixes a and b but does not fix any point in (a, b) .

Lemma 7.9. Let H be a subgroup of F such that $\text{Cl}(H)$ contains the derived subgroup of F . Then there is a dyadic interval $[u] \subseteq (0, 1)$ such that $(0, .u)$ is an orbital of the group $H_{[u]}$.

Proof. Since $[F, F] \leq \text{Cl}(H)$, H acts transitively on the set of finite dyadic fractions \mathcal{D} and in particular, H is not abelian (see Section 11.1 below). Thus, there is a nontrivial element $h \in H \cap [F, F]$. Let b be the minimal number in $(0, 1)$ such that h fixes $[b, 1]$. Then b is finite dyadic and h has an orbital of the form (a, b) for some $a < b$. We will prove that $H_{[b, 1]}$ does not fix any number in $(0, b)$. Then if $[u] \subseteq (0, 1)$ is a dyadic interval with left end-point b , then the lemma holds for $[u]$.

Let u' be a finite binary word such that the right endpoint of $[u']$ is b . Clearly, that is also true for $u'1^k$ for all $k \in \mathbb{N}$. Thus, we can assume that $[u']$ is contained in $(a, b]$. Assume by contradiction that $H_{[b, 1]}$ fixes a point $x \in (0, b)$. Clearly, $x \in (0, a]$. If x is not finite dyadic, we let ω be the unique infinite binary word such that $x = .\omega$. If x is finite dyadic, we let ω be the infinite binary word with a tail of zeros such that $x = .\omega$. Let v be a prefix of ω which contains both digits 0 and 1. We also assume that v is long enough so that $b \notin [v]$. By Corollary 7.8, for some large enough k there is an element f in H with a pair of branches $vw \rightarrow u'1^k$ where w is of length k and vw is a prefix of ω . In particular, $x \in [vw]$. By the choice of ω , x is not the right endpoint of $[vw]$. Notice also that $f(b) > f(.vw1^\mathbb{N}) = .u'1^\mathbb{N} = b$.

We consider the element fhf^{-1} . Since $f(x) \in [u'1^k] \setminus \{.u'1^\mathbb{N}\} \subseteq (a, b)$, we have $h(f(x)) \neq f(x)$. Indeed, (a, b) is an orbital of h . Thus $fhf^{-1}(x) \neq x$. On the other hand, since h fixes the interval $[b, 1]$, the conjugate $h^{f^{-1}}$ fixes the interval $f^{-1}([b, 1]) \supseteq [b, 1]$. Hence $h^{f^{-1}} \in H_{[b, 1]}$, in contradiction to x being a fixed point of $H_{[b, 1]}$. \square

Theorem 7.10. Let H be a subgroup of F . Then H contains the derived subgroup $[F, F]$ if and only if the following 2 conditions are satisfied.

- (1) $\text{Cl}(H)$ contains the derived subgroup of F (equivalently, $\mathcal{L}(H)$ satisfies the conditions of Lemma 7.1).
- (2) There is an element $h \in H$ which fixes a finite dyadic fraction $\alpha \in (0, 1)$ such that $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$.

Proof. It is obvious that if $[F, F] \subseteq H$ then H satisfies the conditions above. Indeed, they are already satisfied for $[F, F]$. Let $H \leq F$ be a subgroup which satisfies the 2 conditions. We will show that $[F, F] \subseteq H$.

Let $h \in H$ be an element satisfying condition (2) in the theorem for some finite dyadic fraction $\alpha \in (0, 1)$. Let $\alpha = .u$ for some finite binary word u ending with 1. By Lemma 2.6(3), h has a pair of branches $u0^m \rightarrow u0^{m-1}$ for some $m \in \mathbb{N}$. Replacing u by $u0^{m-1}$

we get that h has the pair of branches $u0 \rightarrow u$. Notice that this will remain true if u is replaced again by a word $u0^k$ for any k . In particular, we can assume that u contains both digits 0 and 1. Since h fixes a small left neighborhood of α , there is a small enough dyadic interval $[u'] \subseteq (0, 1)$ such that $.u'1^\mathbb{N} = \alpha$ and h fixes the interval $[u']$. Replacing u' with $u'1$ if necessary, we can assume that u' contains both digits 0 and 1. Notice that one can replace u' by any word $u'1^k$ and h would still fix the interval $[u']$.

Let v be the finite binary word such that $[v]$ satisfies the conclusion of Lemma 7.9. Clearly, v contains both digits 0 and 1. Notice that if v is replaced by any word $v0^k$ for some $k \geq 0$, then $(0, .v)$ remains an orbital of $H_{[v]}$. Since u^+ , u'^+ and v^+ contain both digits 0 and 1, by Corollary 7.8, there is an integer k such that for all finite binary words w_1, w_2, w_3 of length $\geq k$, there is an element in H with a pair of branches $u'w_1 \rightarrow uw_2$ and an element in H with a pair of branches $uw_2 \rightarrow vw_3$. We replace u by $u0^k$, u' by $u'1^k$ and v by $v0^k$. In particular, h has a pair of branches $u0 \rightarrow u$ and fixes the interval $[u']$ pointwise. Similarly, $(0, .v)$ is an orbital of $H_{[v]}$. In addition, there is an element $h_t \in H$ with a pair of branches $u \rightarrow v$. The finite binary words u , u' and v and the elements h and h_t will be fixed throughout the proof. Notice that for any pair of finite binary words w'_1, w'_2 there is an element in H with a pair of branches $u'w'_1 \rightarrow uw'_2$ and an element with a pair of branches $u'w'_1 \rightarrow u'w'_2$.

Given a binary word w , we will say that w is *H-equivalent to a 0-extension of v* if for some $m \in \mathbb{N}$, $v0^m \rightarrow w$ is a pair of branches of some element in H . We note that by Corollary 7.8, if w is any finite binary word which contains both digits 0 and 1, then every long enough extension of w is *H-equivalent to a 0-extension of v* . It follows that if $a < b$ are finite dyadic fractions in $(0, 1)$, then there is a subdivision of $[a, b]$ into dyadic intervals $[w_1], \dots, [w_n]$ such that for each i , the word w_i is *H-equivalent to a 0-extension of v* .

We will need the following 3 lemmas.

Lemma 7.11. Let w_1, w_2 be two finite binary words *H-equivalent to a 0-extension of v* . Let $g \in H$ be an element with a pair of branches $w_1 0^{m_1} \rightarrow w_2 0^{m_2}$ for some $m_1, m_2 \geq 0$. Let $a_1 \in (0, .w_1)$ and let $n_1, n_2 \geq 0$. Then there are elements g_ℓ and g_r in H such that g_ℓ and g_r are conjugates of powers of h and the element $g_1 = g_\ell g g_r$ coincides with g on the interval $[a_1, .w_1]$ and has a pair of branches $w_1 0^{n_1} \rightarrow w_2 0^{n_2}$.

Proof. By assumption, there are elements $h_1, h_2 \in H$ such that h_1 has a pair of branches $v0^{k_1} \rightarrow w_1$ and h_2 has a pair of branches $v0^{k_2} \rightarrow w_2$ for some $k_1, k_2 \in \mathbb{N}$. Recall that h_t has a pair of branches $u \rightarrow v$ and let $y = h_t(.u') \in (0, .v)$. Then $h_t([u']) = h_t([.u', .u]) = [y, .v]$. Let $x_1 = h_1^{-1}(a_1)$, then $x_1 < h_1^{-1}(.w_1) = .v$. Since $(0, .v)$ is an orbital of $H_{[v]}$, there is an element $f_1 \in H_{[v]}$ (as such, with a pair of branches $v \rightarrow v$) such that $f_1(y) < x_1$. We consider the element $q_1 = h_t f_1 h_1 \in H$. q_1 has the pair of branches $u0^{k_1} \rightarrow w_1$. Indeed, h_t takes $u0^{k_1}$ onto $v0^{k_1}$, then f_1 takes $v0^{k_1}$ onto itself. Finally, h_1 takes $v0^{k_1}$ onto w_1 . In addition, $q_1([u']) = h_1(f_1(h_t([u']))) = h_1(f_1([y, .v])) \supseteq h_1([x_1, .v]) = [a_1, .w_1]$.

We note that $h^{n_1-m_1}$ has a pair of branches of the form $u0^{n_1} \rightarrow u0^{m_1}$. Indeed, h has a pair of branches of the form $u0 \rightarrow u$. Thus, if $n_1 \geq m_1$ then $h^{n_1-m_1}$ has a pair of branches of the form $u0^{n_1-m_1} \rightarrow u$. One can add a common suffix 0^{m_1} and get that h has the pair of branches $u0^{n_1} \rightarrow u0^{m_1}$. If $n_1 < m_1$, then $h^{n_1-m_1} = (h^{m_1-n_1})^{-1}$ and the result follows from the previous case.

We consider the element $g_\ell = (h^{n_1-m_1})^{q_1} \in H$. From the above, it follows that q_1^{-1} has a pair of branches $w_1 0^{n_1} \rightarrow u 0^{k_1} 0^{n_1}$, $h^{n_1-m_1}$ has a pair of branches $u 0^{n_1} 0^{k_1} \rightarrow u 0^{m_1} 0^{k_1}$ and q_1 has a pair of branches $u 0^{k_1} 0^{m_1} \rightarrow w_1 0^{m_1}$. Thus, g_ℓ has a pair of branches $w_1 0^{n_1} \rightarrow w_1 0^{m_1}$. In addition, since h fixes the interval $[u']$, g_ℓ fixes the interval $q_1([u']) \supseteq [a_1, .w_1]$.

We let $a_2 = g(a_1)$. Clearly, $a_2 \in (0, .w_2)$. As above, one can construct a function $q_2 \in H$ such that q_2 has a pair of branches $u 0^{k_2} \rightarrow w_2$ and such that $q_2([u']) \supseteq [a_2, .w_2]$. Then, the element $g_r = (h^{m_2-n_2})^{q_2} \in H$ has a pair of branches $w_2 0^{m_2} \rightarrow w_2 0^{n_2}$ and g_r fixes the interval $[a_2, .w_2]$.

We let $g_1 = g_\ell g_r$. Then g_1 has the pair of branches $w_1 0^{n_1} \rightarrow w_2 0^{n_2}$. Since g_ℓ fixes the interval $[a_1, .w_1]$ pointwise and g_r fixes the image $g([a_1, .w_1]) = [a_2, .w_2]$ pointwise, the functions g and g_1 coincide on $[a_1, .w_1]$, as required by the lemma. \square

Lemma 7.12. Let $a < b$ be finite dyadic fractions in $(0, 1)$ and let w be a finite binary word with prefix u or u' . Then there is a function $g \in H$ such that $g([a, b]) \subseteq [w]$.

Proof. We consider the proof of Lemma 7.11. As part of the proof, in the notations of Lemma 7.11, we show that there is an element $q_1 \in H$ such that $q_1^{-1}([a_1, .w_1]) \subseteq [u']$. Clearly, for any $b \in (0, 1)$ there are words w_1, w_2 and an element $g \in H$ as in Lemma 7.11 such that $b = .w_1$. Then if one takes $a_1 = a$, one gets that $q_1^{-1}([a, b]) \subseteq [u']$. By the choice of u' and u , since w has prefix u or u' , there is an element $g_1 \in H$ with a pair of branches $u' \rightarrow w$. Thus, $g = q_1^{-1} g_1$ satisfies the result. \square

Lemma 7.13. Let $a < b$ be finite dyadic fractions in $(0, 1)$ and let $f \in [F, F]$. Then there is an element $g_1 \in H \cap [F, F]$ such that g_1 coincides with f on $[a, b]$.

Proof. Since $f \in [F, F]$ it fixes a small neighborhood of 0 and a small neighborhood of 1. Thus, we can choose $a_1 < a$ and $b_1 > b$ in $(0, 1)$ such that f fixes the intervals $[0, a_1]$ and $[b_1, 1]$. Let Δ be a diagram of f . Let $u_i \rightarrow v_i$, $i = 1, \dots, n$ be the pairs of branches of Δ . Replacing Δ by an equivalent diagram if necessary, we can assume that $a_1 \notin [u_1] \cup [u_2]$ and that $b_1 \notin [u_{n-1}] \cup [u_n]$. In particular $u_1 \equiv v_1$, $u_2 \equiv v_2$, $u_{n-1} \equiv v_{n-1}$ and $u_n \equiv v_n$. We can also assume that for all $i \in \{2, \dots, n-1\}$ the words u_i and v_i are H -equivalent to a 0-extension of v .

We start by proving that there exists $g \in H$ such that g is a product of conjugates of h and has the pairs of branches $u_i \rightarrow v_i$ for $i = 2, \dots, n-1$. In particular, it will coincide with f on $[.u_2, .u_n] \supseteq [a, b]$.

Let $f_1 = 1$. We can construct elements f_2, \dots, f_{n-1} inductively so that for every $j \in \{2, \dots, n-1\}$,

- (1) f_j has the pair of branches $u_j \rightarrow v_j$;
- (2) f_j coincides with f_{j-1} on $[.u_2, .u_j]$; and
- (3) $f_j = \ell_j f_{j-1} r_j$ where ℓ_j and r_j are conjugates of powers of h .

Then for $g = f_{n-1}$ we will clearly have the result.

For $j = 2$, since $u_2 \equiv v_2$, we take $f_2 = f_1 = 1$. Clearly, all 3 conditions are satisfied for f_2 .

Now assume that for some $j \in \{2, \dots, n-2\}$, the function f_j was constructed to satisfy the 3 properties above. To construct f_{j+1} we proceed as follows. Δ and f_j have the pair

of branches $u_j \rightarrow v_j$. Since u_j contains the digit 0, we can let p be the prefix of u_j such that $u_j \equiv p01^c$ for some $c \geq 0$. Similarly, let q be the prefix of v_j such that $v_j \equiv q01^d$ for some $d \geq 0$. Then the pair of branches $u_{j+1} \rightarrow v_{j+1}$ of Δ must be of the form $u_{j+1} \equiv p10^{c_1} \rightarrow v_{j+1} \equiv q10^{d_1}$ for some $c_1, d_1 \geq 0$. Similarly, if Δ' is a diagram of f_j with the pair of branches $u_j \rightarrow v_j$, then the next pair of branches in the diagram must be of the form $p10^{c_2} \rightarrow q10^{d_2}$ for some $d_1, d_2 \geq 0$. Let $k = \max\{c_1, d_1\}$. Then by adding the common suffix 0^k , we get that f_j has the pair of branches $p10^{k+c_2} \rightarrow q10^{k+d_2}$; i.e., the pair of branches $u_{j+1}0^{k-c_1+c_2} \rightarrow v_{j+1}0^{k-d_1+d_2}$. Applying Lemma 7.11 with $g = f_j$, $a = .u_2$, $w_1 \equiv u_{j+1}$, $w_2 \equiv v_{j+1}$, $m_1 = k - c_1 + c_2$, $m_2 = k - d_1 + d_2$ and $n_1 = n_2 = 0$, we get that there are elements ℓ_{j+1} and r_{j+1} which are conjugates of powers of h such that the element $\ell_{j+1}f_jr_{j+1}$ has the pair of branches $u_{j+1} \rightarrow v_{j+1}$ and coincides with f_j on the interval $[.u_2, .u_{j+1}]$. We let $f_{j+1} = \ell_{j+1}f_jr_{j+1}$.

Now, let $g = f_{n-1}$. Since g is a product of conjugates of h , we have that $g'(0^+) = (h'(0^+))^l$ and $g'(1^-) = (h'(1^-))^l$ for some $l \in \mathbb{Z}$. By Lemma 7.12 there is an element $h_1 \in H$ such that $h_1([.u_2, .u_n]) \subseteq [u']$.

We let $g_1 = (h^{h_1^{-1}})^{-l}g$. Clearly, $g_1 \in H$. We claim that $g_1 \in [F, F]$ and that g_1 coincides with g (and thus with f) on $[.u_2, .u_n] \supseteq [a, b]$. Since $g'(0^+) = (h'(0^+))^l$, we have that $g_1'(0^+) = 1$. Similarly, $g_1'(1^-) = 1$. Therefore $g_1 \in [F, F]$. Since h fixes the interval $[u']$, $h^{h_1^{-1}}$ fixes the interval $h_1^{-1}([u']) \supseteq [.u_2, .u_n]$, thus g_1 coincides with g on $[.u_2, .u_n]$ as required. \square

Now we are ready to prove the theorem. We consider the dyadic interval $[u]$. The group F is isomorphic to the subgroup $F_{[u]}$ of all functions with support in $[u]$. Let y_0, y_1 be a generating set of $F_{[u]}$. We construct elements $h_0, h_1 \in H \cap [F, F]$ such that

- (1) for $j = 0, 1$, h_j coincides with y_j on $[u]$; and
- (2) the intersection of the support of h_0 and the support of h_1 is contained in $[u]$.

For $j = 0$, by Lemma 7.13, there is an element $h_0 \in H \cap [F, F]$ such that h_0 coincides with y_0 on $[u]$. Since $h_0 \in [F, F]$ there are finite dyadic $a < b$ in $(0, 1)$ such that $[u] \subseteq (a, b)$ and the support of h_0 is contained in (a, b) . We apply Lemma 7.13 to get an element $h_1 \in H \cap [F, F]$ which coincides with y_1 on $[a, b]$. Then conditions (1) and (2) are satisfied for h_0 and h_1 .

Conditions (1) and (2) above imply that the commutator subgroup of $G = \langle h_0, h_1 \rangle$ coincides with the commutator subgroup of $\langle y_0, y_1 \rangle = F_{[u]}$. Therefore, $[F_{[u]}, F_{[u]}] \leq H$.

To prove that $[F, F] \leq H$, we apply Lemma 7.12. If $f \in [F, F]$ then f has support in some interval $[c, d] \subseteq (0, 1)$ for some finite dyadic $c < d$. By Lemma 7.12, there is an element $q \in H$ such that $q([c, d]) \subseteq [u01]$. Then f^q has support in $[u01]$, and in particular, it has support in $[u]$ and slope 1 at both endpoints of $[u]$. It follows that $f^q \in [F_{[u]}, F_{[u]}] \subseteq H$. Then $f \in H^{q^{-1}} = H$ as required. \square

Theorem 7.10 implies the following.

Theorem 7.14. Let H be a subgroup of F . Then $H = F$ if and only if the following conditions are satisfied.

- (1) $\text{Cl}(H)$ contains the derived subgroup of F .
- (2) $H[F, F] = F$

- (3) There is an element $h \in H$ which fixes a finite dyadic fraction α such that $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$.

Proof. If $H = F$ then H clearly satisfies the conditions in the theorem. In the other direction, if H satisfies conditions (1),(3) then by Theorem 7.10, H contains the derived subgroup $[F, F]$. Thus, by condition (2), we have $H = F$. \square

Given a finite number of elements $h_1, \dots, h_n \in F$, it is simple to check if conditions (1) and (2) in the theorem hold for the subgroup H they generate. In the next section we give an algorithm, called the *Tuples algorithm*, for checking if condition (3) of Theorem 7.14 is satisfied, given that condition (1) holds. Thus, we get an algorithm solving the generation problem in Thompson group F .

8 The Tuples algorithm

Let $H \leq F$ be a subgroup of F generated by a finite set X such that $\text{Cl}(H)$ contains the derived subgroup of F . In this section we show that the following problem is decidable.

Problem 8.1. Determine whether there exists an element $h \in H$ which has a dyadic break point α such that the slope of h at α^- is 1 and the slope of h at α^+ is 2.

Lemma 8.2. Let $H \leq F$ be a subgroup such that $\text{Cl}(H)$ contains the derived subgroup of F . Then the following are equivalent.

1. There is an element $h_1 \in H$ which fixes a finite dyadic fraction α_1 such that $h'_1(\alpha_1^-) = 1$ and $h'_1(\alpha_1^+) = 2$.
2. There is an element $h_2 \in H$ which fixes a finite dyadic fraction α_2 such that $h'_2(\alpha_2^-) = 2$ and $h'_2(\alpha_2^+) = 1$.

Proof. We show that (2) implies (1). The converse implication is similar. By Lemma 7.1, $(010)^+ = (01)^+$ in the core $\mathcal{L}(H)$. Thus, by Lemma 4.6 for a large enough k there is an element $h \in H$ with a pair of branches $010^{k+1} \rightarrow 010^k$. In particular, $\alpha = .01$ is a fixed point of h and the slope $h'(\alpha^+) = 2$.

Since $\text{Cl}(H)$ contains $[F, F]$, the action of $\text{Cl}(H)$, and thus, of H , on the set of finite dyadic fractions \mathcal{D} is transitive. Thus, there is an element $g \in H$ such that $g(\alpha_2) = \alpha$. We consider the element $f = h_2^g$. Since α_2 is a fixed point of h_2 , α is a fixed point of f . Similarly, $f'(\alpha^-) = 2$ and $f'(\alpha^+) = 1$.

Since $h \in F$, the slope $h'(\alpha^-) = 2^m$ for some $m \in \mathbb{Z}$. We consider the element $h_1 = hf^{-m} \in H$. Clearly, h_1 fixes α . In addition $h'_1(\alpha^-) = 2^m \cdot 2^{-m} = 1$ and $h'_1(\alpha^+) = 2$. Thus, h_1 is an element satisfying condition (1) for $\alpha_1 = \alpha$. \square

Definition 8.3. Let $H \leq F$ be a subgroup of F . We denote by \mathcal{S}_H the subset of \mathbb{Z}^2 of all vectors (a, b) such that there is an element $h \in H$ and a finite dyadic fraction $\alpha \in (0, 1)$ such that h fixes α , $h'(\alpha^-) = 2^a$ and $h'(\alpha^+) = 2^b$.

It is obvious that if H acts transitively on \mathcal{D} then \mathcal{S}_H is a subgroup of \mathbb{Z}^2 . Lemma 8.2 implies the following.

Corollary 8.4. Let $H \leq F$ be a subgroup such that $\text{Cl}(H)$ contains the derived subgroup of F , then $(0, 1) \in \mathcal{S}_H$ if and only if $\mathcal{S}_H = \mathbb{Z}^2$.

Thus, to solve Problem 8.1 it suffices to determine if $\mathcal{S}_H = \mathbb{Z}^2$. For the rest of this section we fix a finitely generated subgroup $H \leq F$ and a generating set $X = \{g_1, \dots, g_n\}$. We assume that $\text{Cl}(H)$ contains $[F, F]$. By Lemma 7.1, every finite binary word u labels a path on $\mathcal{L}(H)$.

Definition 8.5 (The equivalence relation R_X). We define an equivalence relation R_X on the set of all finite binary words \mathcal{B} (such that $\emptyset \in \mathcal{B}$). Let $\mathcal{L}_{\text{sem}}(X)$ be the semi-core of H associated with the generating set X , when diagrams in X are taken in reduced form (see Section 7). Two finite binary words u and v are said to be R_X -equivalent if $u^+ = v^+$ in $\mathcal{L}_{\text{sem}}(X)$. We write $u \sim_X v$ and denote the equivalence class of u in R_X by $[u]_X$.

By Lemma 7.5, if $u \sim_X v$ then there is an element $h \in H$ with a pair of branches $u \rightarrow v$. Note also that the number of equivalence classes in R_X is finite (and computable). Indeed, it is equal to the number of edges in $\mathcal{L}_{\text{sem}}(X)$. We remark that $[\emptyset]_X = \{\emptyset\}$.

Let Ψ be a tree-diagram over \mathcal{K} . Recall that by Remark 2.5, if u_1 and u_2 are consecutive branches of Ψ and u is the longest common prefix of u_1 and u_2 , then $u_1 \equiv u01^m$ and $u_2 \equiv u10^n$ for some $m, n \geq 0$.

Definition 8.6 (Tuples associated with a diagram in H). Let Δ be a diagram of an element in H . Let u_1 and u_2 be a pair of consecutive positive branches of Δ and v_1 and v_2 be the corresponding pair of consecutive negative branches of Δ , so that $u_1 \rightarrow v_1$ and $u_2 \rightarrow v_2$ are pairs of branches of Δ . Let u be the longest common prefix of u_1 and u_2 . By Remark 2.5,

$$u_1 \equiv u01^{m_1} \text{ and } u_2 \equiv u10^{n_1} \text{ for some } m_1, n_1 \geq 0.$$

Let v be the longest common prefix of v_1 and v_2 . By Remark 2.5,

$$v_1 \equiv v01^{m_2} \text{ and } v_2 \equiv v10^{n_2} \text{ for some } m_2, n_2 \geq 0.$$

We define the *tuple associated with the consecutive pairs of branches $u_1 \rightarrow v_1$ and $u_2 \rightarrow v_2$ of the diagram Δ* to be the tuple

$$(m_1 - m_2, n_1 - n_2, [u]_X \rightarrow [v]_X),$$

where $[u]_X$ and $[v]_X$ are the equivalence classes of u and v in R_X . The tuple can be viewed as an element of $\mathbb{Z} \times \mathbb{Z} \times (R_X / \sim_X \times R_X / \sim_X)$.

Usually, we will refer to tuples as tuples associated with a diagram without mentioning the consecutive pairs of branches.

Definition 8.7 (The groupoid \mathcal{T}_H). We define the set \mathcal{T}_H to be the set of all tuples associated with diagrams of elements in H . We define two operations on tuples in \mathcal{T}_H as follows.

Taking inverse: For a tuple $t = (a, b, [u]_X \rightarrow [v]_X)$ in \mathcal{T}_H we define the inverse tuple

$$t^{-1} = (-a, -b, [v]_X \rightarrow [u]_X).$$

(Partial) addition: Given two tuples $(a, b, [u]_X \rightarrow [v]_X)$ and $(c, d, [v]_X \rightarrow [w]_X)$, we let

$$(a, b, [u]_X \rightarrow [v]_X) + (c, d, [v]_X \rightarrow [w]_X) = (a + c, b + d, [u]_X \rightarrow [w]_X).$$

The following lemma shows that \mathcal{T}_H is closed under the operations of taking inverses and addition. It follows easily that \mathcal{T}_H is a groupoid.

Lemma 8.8. The set \mathcal{T}_H is closed under taking inverses and addition.

Proof. It is obvious that \mathcal{T}_H is closed under taking inverses. Indeed, if $t = (a, b, [u]_X \rightarrow [v]_X) \in \mathcal{T}_H$, then the tuple t is associated with a pair of consecutive branches of a diagram Δ in H . Then t^{-1} is associated with the corresponding pair of consecutive branches of the diagram $\Delta^{-1} \in H$.

Let $t_1 = (a, b, [u]_X \rightarrow [v]_X)$ and $t_2 = (c, d, [v]_X \rightarrow [w]_X)$ be tuples in \mathcal{T}_H . The tuple t_1 belonging to \mathcal{T}_H implies that there is a diagram Δ_1 of an element in H which has consecutive pairs of branches

$$u_1 01^{m_1} \rightarrow v_1 01^{m_2} \quad \text{and} \quad u_1 10^{n_1} \rightarrow v_1 10^{n_2}$$

such that $m_1 - m_2 = a$, $n_1 - n_2 = b$, $u_1 \in [u]_X$ and $v_1 \in [v]_X$. Similarly, there is a diagram Δ_2 with consecutive pairs of branches

$$v_2 01^{k_1} \rightarrow w_2 01^{k_2} \quad \text{and} \quad v_2 10^{l_1} \rightarrow w_2 10^{l_2},$$

where $k_1 - k_2 = c$, $l_1 - l_2 = d$, $v_2 \in [v]_X$ and $w_2 \in [w]_X$.

We can assume that $m_2 = k_1$ and $n_2 = l_1$. Indeed, if $m_2 < k_1$, we consider the edge e on the horizontal 1-path of Δ_1 which is the common terminal edge of the positive branch $u_1 01^{m_1}$ and the negative branch $v_1 01^{m_2}$. We replace the edge e with the diagram of the identity with branches $b \rightarrow b$ for all $b \in \{0, 1\}^{k_1 - m_2}$. The resulting diagram is equivalent to Δ_1 and has consecutive pairs of branches $u_1 01^{m_1 + k_1 - m_2} \rightarrow v_1 01^{k_1}$ and $u_1 10^{n_1} \rightarrow v_1 10^{n_2}$. Thus, one can replace m_1 with $m_1 + k_1 - m_2$ and m_2 with k_1 . In a similar way, one can treat the case where $m_2 > k_1$ or $n_2 \neq l_1$. Thus, we can assume that $m_2 = k_1$ and $n_2 = l_1$.

Since $v_1, v_2 \in [v]_X$, there is an element $h \in H$ with a pair of branches $v_1 \rightarrow v_2$. (If $v = \emptyset$, we take h to be the identity.) Let h_1 be the element of H represented by Δ_1 and let h_2 be the element represented by Δ_2 . We consider the element $g = h_1 h h_2$. g has the following consecutive pairs of branches $u_1 01^{m_1} \rightarrow w_2 01^{k_2}$ and $u_1 10^{n_1} \rightarrow w_2 10^{l_2}$. It suffices to note that $m_1 - k_2 = m_1 - m_2 + k_1 - k_2 = a + c$, $n_1 - l_2 = n_1 - n_2 + l_1 - l_2 = b + d$, $u_1 \in [u]_X$ and $w_2 \in [w]_X$. Thus, the tuple

$$t_1 + t_2 = (a + c, b + d, [u]_X \rightarrow [w]_X) \in \mathcal{T}_H.$$

□

For a finite binary word u , we let $\mathcal{T}_H([u]_X)$ be the set of all tuples in \mathcal{T}_H such that the last coordinate is of the form $[u]_X \rightarrow [u]_X$. Such tuples are called *spherical tuples*. Clearly, for each u , $\mathcal{T}_H([u]_X)$ is a commutative group with neutral element $(0, 0, [u]_X \rightarrow [u]_X)$.

We let $\Psi: \mathcal{T}_H \rightarrow \mathbb{Z}^2$ be the natural homomorphism such that

$$\Psi((a, b, [u]_X \rightarrow [v]_X)) = (a, b).$$

Under this homomorphism, each group $\mathcal{T}_H([u]_X)$ embeds into \mathbb{Z}^2 .

Lemma 8.9. Let u and v be finite binary words. Then the groups $\mathcal{T}_H([u]_X)$ and $\mathcal{T}_H([v]_X)$ are conjugate in the groupoid \mathcal{T}_H .

Proof. We consider the finite binary words $u' \equiv u01$ and $v' \equiv v01$. Since u' and v' contain both digits 0 and 1 and $\text{Cl}(H)$ contains $[F, F]$, $(u')^+$ and $(v')^+$ coincide in $\mathcal{L}(H)$ (see Lemma 7.1). By Lemma 4.6 for k large enough there is an element $h \in H$ with a pair of branches $u'1^k \equiv u01^{k+1} \rightarrow v'1^k \equiv v01^{k+1}$. Let Δ be a diagram of h with this pair of branches. The following pair must be of the form $u10^{m_1} \rightarrow v10^{m_2}$ for some $m_1, m_2 \geq 0$. Thus, the tuple

$$t = ((k+1) - (k+1), m_1 - m_2, [u]_X \rightarrow [v]_X) = (0, m, [u]_X \rightarrow [v]_X) \in \mathcal{T}_H$$

for $m = m_1 - m_2$. Conjugating $\mathcal{T}_H([u]_X)$ by the tuple t gives the group $\mathcal{T}_H([v]_X)$. \square

Corollary 8.10. Let u and v be finite binary words. Then

$$\Psi(\mathcal{T}_H([u]_X)) = \Psi(\mathcal{T}_H([v]_X)).$$

Lemma 8.11. Let u be a finite binary word. Then

$$\Psi(\mathcal{T}_H([u]_X)) = \mathcal{S}_H.$$

Proof. Recall that \mathcal{S}_H is the subgroup of \mathbb{Z}^2 of all vectors (a, b) for which there is an element $h \in H$ which fixes a finite dyadic fraction α such that $h'(\alpha^-) = 2^a$ and $h'(\alpha^+) = 2^b$.

Let $t = (a, b, [u]_X \rightarrow [u]_X) \in \mathcal{T}_H([u]_X)$. To prove that $(a, b) \in \mathcal{S}_H$, we consider a diagram Δ of an element h_1 in H with which t is associated. In particular, Δ has consecutive pairs of branches of the form

$$u_1 01^{m_1} \rightarrow u_2 01^{m_2} \quad \text{and} \quad u_1 10^{n_1} \rightarrow u_2 10^{n_2}$$

such that $m_1 - m_2 = a$, $n_1 - n_2 = b$ and $u_1, u_2 \in [u]_X$. Since $u_1 \sim_X u_2$ there is an element $h_2 \in H$ with a pair of branches $u_2 \rightarrow u_1$. Let $h = h_1 h_2$. Then h has consecutive pairs of branches

$$u_1 01^{m_1} \rightarrow u_1 01^{m_2} \quad \text{and} \quad u_1 10^{n_1} \rightarrow u_1 10^{n_2}.$$

In particular, h fixes $\alpha = .u_1 1$. In addition $h'(\alpha^-) = 2^{m_1 - m_2} = 2^a$ and $h'(\alpha^+) = 2^{n_1 - n_2} = 2^b$. Thus $(a, b) \in \mathcal{S}_H$.

In the other direction, let $(a, b) \in \mathcal{S}_H$. Let $h \in H$ be an element which fixes a finite dyadic fraction $\alpha \in (0, 1)$ such that $h'(\alpha^-) = 2^a$ and $h'(\alpha^+) = 2^b$. In particular, in a small enough left (resp. right) neighborhood of α , the slope of h is 2^a (resp. 2^b). Let v be a finite binary word ending with the digit 1 so that $\alpha = .v$. Let v' be the prefix of v such that $v \equiv v'1$. For all $k \geq 0$, the interval $[v'01^k]$ is a left neighborhood of α . For large enough $k > a$, the interval $[v'01^k]$ is a small enough left neighborhood of α so that h has slope 2^a on the interval. Since h fixes α , the interval $[v'01^k]$ is mapped linearly onto $[v'01^{k-a}]$. In other words, h has the pair of branches $v'01^k \rightarrow v'01^{k-a}$. Let Δ be a diagram of h which has this pair of branches. Clearly, the following pair must be of the form $v'10^{m_1} \rightarrow v'10^{m_2}$, for some $m_1, m_2 \geq 0$. Since $h'(\alpha^+) = 2^b$, we have $m_1 - m_2 = b$. Thus, the tuple

$$t = (k - (k - a), m_1 - m_2, [v']_X \rightarrow [v']_X) = (a, b, [v']_X \rightarrow [v']_X) \in \mathcal{T}_H([v']_X).$$

It follows that $(a, b) \in \Psi(\mathcal{T}_H([v']_X)) = \Psi(\mathcal{T}_H([u]_X))$, by Corollary 8.10. \square

To determine whether $\mathcal{S}_H = \mathbb{Z}^2$, it suffices to find a finite generating set M of \mathcal{S}_H . We start by choosing a generating set of \mathcal{T}_H .

Recall that $X = \{g_1, \dots, g_n\}$ is the fixed generating set of H . For each i , we let Δ_i be the reduced diagram of g_i . We let Y be the set of all tuples in \mathcal{T}_H associated with consecutive pairs of branches of the diagrams $\Delta_i^{\pm 1}$. For each equivalence class $[u]_X$, we add to Y the tuple $0_{[u]_X} = (0, 0, [u]_X \rightarrow [u]_X)$. (Notice that all tuples $0_{[u]_X} \in \mathcal{T}_H$. Indeed, one can consider a diagram of the identity element of F with consecutive pairs of branches of the form $u0 \rightarrow u0$ and $u1 \rightarrow u1$.) To prove that Y is a generating set of \mathcal{T}_H , we will need the following two lemmas. The proof of Lemma 8.12 is simple and is left as an exercise to the reader.

Lemma 8.12. Let Δ be a diagram of an element in H . Let $u \rightarrow v$ be a pair of branches of Δ . Let Δ' be the diagram resulting by replacing the edge on the horizontal 1-path of Δ at the end of the positive branch u by a dipole of type 1. Then the tuples in \mathcal{T}_H corresponding to consecutive pairs of branches of Δ' are exactly the tuples associated with Δ and the tuple $(0, 0, [u]_X \rightarrow [v]_X)$. \square

Lemma 8.13. Let $h \in H$. Then h has a diagram Δ which satisfies the following conditions.

- (1) For each pair of branches $u \rightarrow v$ of Δ , we have $[u]_X = [v]_X$.
- (2) All the tuples in \mathcal{T}_H associated with the diagram Δ belong to the sub-groupoid of \mathcal{T}_H generated by Y .

Proof. Before proving the lemma we make the observation that if a diagram Δ satisfies the conditions in the lemma and Δ' results from Δ by the replacement of an edge on the horizontal 1-path of Δ by a dipole of type 1, then Δ' also satisfies the conditions in the lemma. Indeed, inserting the dipole means replacing a pair of branches $u_1 \rightarrow v_1$ by two pairs of branches $u_1 0 \rightarrow v_1 0$ and $u_1 1 \rightarrow v_1 1$. Since $[u_1]_X = [v_1]_X$ implies that $[u_1 0]_X = [v_1 0]_X$ and $[u_1 1]_X = [v_1 1]_X$ (indeed, no foldings of type 1 are applicable to $\mathcal{L}_{\text{sem}}(X)$), condition (1) of the lemma is satisfied for Δ' . Conditions (1) and (2) for Δ and Lemma 8.12 imply that condition (2) of the lemma is satisfied for Δ' .

To prove the lemma we use induction on the word-length m of h with respect to the generating set X . If $m = 1$, then $h = g_i^{\pm 1}$ and one can take the reduced diagram Δ_i or its inverse. By the definition of $\mathcal{L}_{\text{sem}}(X)$, condition (1) is satisfied. Condition (2) is clearly satisfied by the definition of the set Y .

Assume that the lemma is satisfied for every element of H of word length smaller than m and let h be an element of word length m . Then $h = f g_i^{\pm 1}$ where $f \in H$ is an element of word-length $m - 1$ and $g_i \in X$. We assume that $h = f g_i$. The proof in the other case is similar. Let Δ be a diagram for f which satisfies both conditions in the lemma. The reduced diagram Δ_i of the generator g_i also satisfies the conditions. By inserting dipoles of type 1 to Δ and Δ_i , one can get equivalent diagrams Δ' and Δ'_i such that $\Delta'^- \equiv ((\Delta'_i)^+)^{-1}$. Then $\Delta \Delta_i = \Delta' \Delta'_i = \Delta'^+ \circ \Delta'^-_-$ is a diagram of the element h . We denote $\Delta'^+ \circ \Delta'^-_-$ by Δ_h .

We note that if $u \rightarrow v$ is a pair of branches of Δ_h , then for some binary word w , $u \rightarrow w$ is a pair of branches of Δ' and $w \rightarrow v$ is a pair of branches of Δ'_i . Since Δ' and Δ'_i satisfy condition (1), $[u]_X = [w]_X = [v]_X$ and condition (1) is satisfied for the diagram Δ_h .

To see that every tuple in \mathcal{T}_H associated with the diagram Δ_h belongs to the sub-groupoid generated by Y , it is enough to observe that the tuple associated with the i and $i + 1$ pairs

of branches of Δ_h is the sum of the tuple associated with the i and $i + 1$ pairs of branches of Δ' and the tuple associated with the i and $i + 1$ pairs of branches of Δ'_i . Then the result follows from condition (2) for the diagrams Δ' and Δ'_i . \square

Lemma 8.14. The set Y generates the groupoid \mathcal{T}_H .

Proof. Let Δ be a diagram of an element h in H . It suffices to show that all tuples in \mathcal{T}_H associated with Δ belong to the sub-groupoid of \mathcal{T}_H generated by Y . By Lemma 8.13, h can be represented by a diagram Δ' such that

- (1) for every pair of branches $u \rightarrow v$ of Δ' we have $[u]_X = [v]_X$;
- (2) all the tuples in \mathcal{T}_H associated with Δ' belong to the sub-groupoid generated by Y .

There is a diagram Δ'' equivalent to both Δ and Δ' which results from Δ and from Δ' by insertions of dipoles of type 1. It follows from the proof of Lemma 8.13 that all tuples associated with Δ'' also satisfy conditions (1) and (2) above and in particular, they also belong to the sub-groupoid generated by Y . Since Δ'' results from Δ by insertion of dipoles of type 1, it follows from Lemma 8.12, that all tuples associated with Δ are also associated with Δ'' and as such they all belong to the sub-groupoid $\langle Y \rangle$ as required. \square

Let N be the number of equivalence classes of the relation R_X (i.e., the number of distinct edges in the semi-core $\mathcal{L}_{\text{sem}}(X)$). Let M' be the set of all spherical tuples in \mathcal{T}_H of word length at most N with respect to the generating set Y . Let $M = \Psi(M')$. Clearly, the set M is a finite subset of \mathcal{S}_H .

Lemma 8.15. The set M is a generating set of \mathcal{S}_H .

Proof. Let $(a, b) \in \mathcal{S}_H$. We claim that (a, b) belongs to $\langle M \rangle$. Let u be a finite binary word. By Lemma 8.11, the tuple $t = (a, b, [u]_X \rightarrow [u]_X)$ belongs to $T_H([u]_X)$. By Lemma 8.14, t is a product of tuples

$$t = (a_1, b_1, [v_1]_X \rightarrow [v_2]_X) \cdots (a_m, b_m, [v_m]_X \rightarrow [v_{m+1}]_X)$$

where all tuples $(a_i, b_i, [v_i]_X \rightarrow [v_{i+1}]_X)$ belong to Y . Clearly, $[v_1]_X = [v_{m+1}]_X = [u]_X$. We prove the lemma by induction on m . If $m \leq N$, then $t \in M'$. Then $(a, b) \in M$ and we are done. If $m > N$, then for some $i < j$ in $\{1, \dots, m\}$ we have $[v_i]_X = [v_j]_X$. Let $[v]_X = [v_i]_X = [v_j]_X$. We let

$$t' = (a_i, b_i, [v_i]_X \rightarrow [v_{i+1}]_X) \cdots (a_{j-1}, b_{j-1}, [v_{j-1}]_X \rightarrow [v_j]_X) \in \mathcal{T}_H([v]_X).$$

Then $t' = (a', b', [v]_X \rightarrow [v]_X)$ for some $a', b' \in \mathbb{Z}$. By induction, $\Psi(t') = (a', b') \in \langle M \rangle$. It remains to observe that

$$(a, b) = \Psi(t) = \Psi(t') \Psi((a_1, b_1, [v_1]_X \rightarrow [v_2]_X) \cdots (a_{i-1}, b_{i-1}, [v_{i-1}]_X \rightarrow [v_i]_X) \\ (a_j, b_j, [v_j]_X \rightarrow [v_{j+1}]_X) \cdots (a_m, b_m, [v_m]_X \rightarrow [v_{m+1}]_X))$$

and apply the induction hypothesis. \square

Notice that Lemma 8.15 provides an algorithm for the solution of Problem 8.1. Indeed, given a finite subset X of F , one can construct the finite generating set M of \mathcal{S}_H . Then determining whether M generates \mathbb{Z}^2 is a simple linear algebra problem.

9 F is a cyclic extension of a subgroup K which has a maximal elementary amenable subgroup

In this section we apply the methods developed in this paper to prove the following.

Theorem 9.1. There is a chain of subgroups $B \leq K \leq F$ in Thompson group F such that

- (1) K is a normal subgroup of F and the quotient F/K is infinite cyclic.
- (2) B is a maximal subgroup of K . Moreover, for any $f \in F \setminus B$, we have $K \leq \langle B, f \rangle$.
- (3) B is elementary amenable, $\text{Cl}(B) = B$ and the action of B on the set of finite dyadic fractions \mathcal{D} is transitive.

In particular, B from Theorem 9.1 is an elementary amenable subgroup of F such that the lattice of subgroups of F strictly containing B is isomorphic to the lattice of subgroups of \mathbb{Z} . It is obvious that K and F are co-amenable. In fact, since K contains the derived subgroup of F , it contains many copies of F .

Proof of Theorem 9.1. In [8, Section 5], Brin defines an elementary amenable group G_1 of elementary class $\omega + 2$. The same group was defined independently about the same time by Navas [23, Example 6.3]. To realize G_1 as a subgroup of F , it suffices to let x be an element of F with a single orbital (a, b) and let y be a function in F which maps (a, b) into a fundamental domain of x . Then the group $\langle x, y \rangle$ is a copy of G_1 in F .

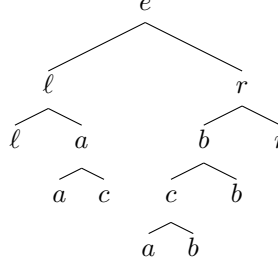
We let $x = x_0 x_1 x_2^{-1} x_0^{-1}$ and $y = x_0 x_1^{-2}$ and take B to be the subgroup of F generated by x and y . The pairs of branches of the reduced diagrams of x and y are as follows.

$$x = \begin{cases} 00 & \rightarrow 00 \\ 010 & \rightarrow 01 \\ 011 & \rightarrow 100 \\ 10 & \rightarrow 101 \\ 11 & \rightarrow 11 \end{cases} \quad y = \begin{cases} 00 & \rightarrow 0 \\ 01 & \rightarrow 1000 \\ 10 & \rightarrow 1001 \\ 110 & \rightarrow 101 \\ 111 & \rightarrow 11 \end{cases}$$

Notice that x has a single orbital $(.01, .11)$. The function y maps $(.01, .11)$ onto $(.1, .101)$. Since $x(.1) = .101$, we have that $y(.01, .11)$ is contained in a single fundamental domain of x . In particular, B is isomorphic to the group G_1 and as such it is elementary amenable.

We let $K = B[F, F]$. Then K is a normal subgroup of F . We note that $\pi_{\text{ab}}(K) = \pi_{\text{ab}}(B) = \langle (1, -2) \rangle \leq \mathbb{Z}^2$. Since \mathbb{Z}^2 is a cyclic extension of $\pi_{\text{ab}}(K)$, F is a cyclic extension of K . Thus, condition (1) of the theorem is satisfied.

To prove condition (3), we consider the core $\mathcal{L}(B)$. It can be described by the following binary tree.



To prove that B acts transitively on \mathcal{D} we note that every edge in $\mathcal{L}(B)$ is the top edge of some positive cell. The edges of $\mathcal{L}(B)$ which are not incident to $\iota(\mathcal{L}(B))$ are a, b, c and r . Since $\iota(r) = \iota(b) = \iota(c) = \iota(a)$, there is a unique inner vertex in $\mathcal{L}(B)$. Hence, by Corollary 6.9, B acts transitively on the set \mathcal{D} .

By Lemma 3.8, $\text{Cl}(B)$ is naturally isomorphic to the diagram group $\text{DG}(\mathcal{L}(B), e)$. Applying the algorithm from [17, Lemma 9.11] for finding a generating set of a diagram group (over a “nice enough” semigroup presentation), one can show that the generating set $\{x, y\}$ of B is also a generating set of $\text{Cl}(B)$. Hence, B is a closed subgroup of F . Thus, condition (3) holds for B .

To prove that condition (2) of Theorem 9.1 is satisfied, we make use of the following lemma.

Lemma 9.2. Let $f \in F \setminus B$ and let H be the subgroup of F generated by $B \cup \{f\}$. Then $\text{Cl}(H)$ contains the derived subgroup of F (in fact, $\text{Cl}(H) = F$).

Proof. We observe that every edge in $\mathcal{L}(B)$ is the top edge of some positive cell. Thus, every finite binary word u labels a path on $\mathcal{L}(B)$. We also note that as $\mathcal{L}(B)$ has a unique left boundary edge ℓ , every path $u \equiv 0^n$ for $n \in \mathbb{N}$ must terminate on ℓ . Similarly, every path 1^m for $m \in \mathbb{N}$ terminates on the edge r .

Now let Δ be the reduced diagram of f . Since Δ is not accepted by $\mathcal{L}(B)$ (indeed, $f \notin B = \text{Cl}(B)$), Δ must have a pair of branches $u_1 \rightarrow v_1$, such that on $\mathcal{L}(B)$, u_1^+ and v_1^+ are distinct edges. It is obvious that both u_1 and v_1 must contain both digits 0 and 1. Thus, u_1^+, v_1^+ are inner edges of $\mathcal{L}(B)$ (see Corollary 4.10).

Next, we consider the core $\mathcal{L}(H)$. There is a natural morphism ψ from the core $\mathcal{L}(B)$ to the core $\mathcal{L}(H)$. Indeed, to construct the core of H , one can start with the core $\mathcal{L}(B)$, and the diagram Δ ; attach the top and bottom edges of Δ to the distinguished edge of $\mathcal{L}(B)$ and apply foldings. Since every finite binary word u labels a path on $\mathcal{L}(B)$, each edge (resp. cell) of the attached diagram Δ would be folded onto some edge (resp. cell) of $\mathcal{L}(B)$. The morphism ψ maps an edge e' (resp. cell π) of $\mathcal{L}(B)$ to the edge (resp. cell) of $\mathcal{L}(H)$, identified with the edge e' (resp. cell π) in this process. It is obvious that ψ is surjective and that ψ maps inner (resp. boundary) edges to inner (resp. boundary) edges. Thus, the inner edges of $\mathcal{L}(H)$ are $\psi(a), \psi(b)$ and $\psi(c)$. Every path on $\mathcal{L}(B)$ is mapped by ψ to a path on $\mathcal{L}(H)$. In particular, any finite binary word w labels a path on $\mathcal{L}(H)$.

We claim that $\mathcal{L}(H)$ has a unique inner edge. To prove that, we consider the options for the pair of edges u_1^+, v_1^+ in $\mathcal{L}(B)$. If $\{u_1^+, v_1^+\} = \{a, b\}$, then Δ being accepted by $\mathcal{L}(H)$ implies that in $\mathcal{L}(H)$, the paths u_1 and v_1 terminate on the same edge. Hence, $\psi(a) = \psi(b)$. Notice that $\psi(a)$ is the top edge of a cell π_1 in $\mathcal{L}(H)$ with bottom path $\psi(a)\psi(c)$. Similarly,

$\psi(b)$ is the top edge of a cell π_2 with bottom path $\psi(c)\psi(b)$. Since $\psi(a) = \psi(b)$ and no foldings are applicable to $\mathcal{L}(H)$, we must have $\pi_1 = \pi_2$ and thus, the left bottom edges of the cells satisfy $\psi(a) = \psi(c)$. Thus, $\psi(a) = \psi(b) = \psi(c)$ is the only inner edge of $\mathcal{L}(H)$. A similar argument for the case where $\{u_1^+, v_1^+\} = \{b, c\}$ or $\{u_1^+, v_1^+\} = \{a, c\}$ shows that $\mathcal{L}(H)$ has a unique inner edge. By Lemma 7.1, $\text{Cl}(H)$ contains the derived subgroup of F . \square

Now we can finish the proof using a simple application of Theorem 7.10. We note that for $x \in B$ and $\alpha = .01$ we have $x'(\alpha^+) = 2$ and $x'(\alpha^-) = 1$. By Lemma 9.2, for any $f \notin B$, the closure of $\langle B, f \rangle$ contains the derived subgroup of F . Hence, by Theorem 7.10, for any $f \notin B$, $\langle B, f \rangle$ contains $[F, F]$. It follows that $K \leq \langle B, f \rangle$. \square

Remark 9.3. The group $K = B[F, F]$ is 2-generated. Indeed, one can show that it is generated by $x_0x_1^{-2}$ and $x_1x_3^{-1}$ by another application of Theorem 7.10. If $K' = \langle x_0x_1^{-2}, x_1x_3^{-1} \rangle$ then by considering the image in the abelianization we get that $K'[F, F] = K$. One can check that $\text{Cl}(K') = F$. Then since the element $x_0x_1^{-2}$ fixes the fraction $.1$, has slope 1 at $.1^-$ and slope 2 at $.1^+$, Theorem 7.10 implies that $[F, F] \leq K'$ and as such $K = K'$.

10 Computations related to $\mathcal{L}(H)$

10.1 On the algorithm for finding a generating set of $\text{Cl}(H)$

Let \mathcal{K}' be a directed 2-complex. (We denote it by \mathcal{K}' , as \mathcal{K} is still used to denote the Dunces hat.) As noted in Section 2.2.A, \mathcal{K}' can be described in a form similar to a semigroup presentation. We let

$$\mathcal{P} = \langle E \mid \mathbf{top}(f) \rightarrow \mathbf{bot}(f), f \in F^- \rangle$$

be the *semigroup presentation associated with \mathcal{K}'* . We use negative 2-cells instead of positive ones, as it would be more convenient below. The presentation \mathcal{P} defines a semigroup S associated with the directed 2-complex \mathcal{K}' . The semigroup S is closely related to the diagram groupoid $\mathcal{D}(\mathcal{K}')$. Indeed, let u and v be two 1-paths in \mathcal{K}' . Then u and v can be viewed as words over the alphabet E . Then u and v represent the same element of S if and only if there is a (u, v) -diagram over \mathcal{K}' [19]. In particular, if $u = v$ in S , then $\iota(u) = \iota(v)$ and $\tau(u) = \tau(v)$. In general, if u and v are two words in the alphabet E , then u and v are equal as elements of S if and only if there is a (u, v) -diagram over \mathcal{K}'' , where \mathcal{K}'' is the directed 2-complex resulting from \mathcal{K}' by the identification of all vertices to a single vertex [19]. We also note that if words u and v are equal in S then u is a 1-path in \mathcal{K}' if and only if v is a 1-path in \mathcal{K}' .

As mentioned in the proof of Theorem 9.1, there is an algorithm due to Guba and Sapir [17] for finding the generating set of a diagram group. Let p be a 1-path in \mathcal{K}' . To find a generating set for $\text{DG}(\mathcal{K}', p)$ we make some further assumptions about the directed 2-complex \mathcal{K}' . First, we assume that the set of edges E is equipped with a total-order \prec . The total order \prec induces a lexicographic order on the set E^* of words in the alphabet E . If w_1 and w_2 are two words over E then w_1 is smaller than w_2 in the *ShortLex* order if $|w_1| < |w_2|$ or $|w_1| = |w_2|$ and w_1 precedes w_2 in the lexicographic order. We assume that for each positive 2-cell f in \mathcal{K}' , $\mathbf{top}(f)$ is smaller than $\mathbf{bot}(f)$ in the ShortLex order. In particular, in each rewriting rule $r_1 \rightarrow r_2$ in the presentation \mathcal{P} , r_2 is smaller than r_1 in the ShortLex order.

Let \mathcal{P}' be a completion of \mathcal{P} (for terminology see [25]) such that every rewriting rule $r_1 \rightarrow r_2$ in \mathcal{P}' satisfies $r_1 > r_2$ in the ShortLex order. We also assume that for any relation $r_1 \rightarrow r_2$ in \mathcal{P}' we can point on a derivation over \mathcal{P} from r_1 to r_2 . Notice that if one applies the Knuth-Bendix algorithm (see, for example, [25]) for finding a completion \mathcal{P}' of \mathcal{P} , then the resulting completion (if attained) satisfies these 2 properties. Whenever a presentation \mathcal{P} associated with a directed 2-complex and a completion \mathcal{P}' of it are mentioned below, we assume that the set E is equipped with a total order $<$ and that \mathcal{P} and \mathcal{P}' satisfy the properties mentioned above.

To implement the algorithm from [17, Lemma 9.11] for finding a generating set of $\text{DG}(\mathcal{K}', p)$ one has to find the set B of all tuples $[u, r_1 \rightarrow r_2, v]$ such that

- (1) u and v are words in the alphabet E which are reduced over the complete presentation \mathcal{P}' ;
- (2) $r_1 \rightarrow r_2$ is a rewriting rule of \mathcal{P}' ;
- (3) ur_1v and p are equal as elements of S ; and
- (4) In the notations of [17], if one lets $u \equiv u$, $v \equiv v$, $r \equiv r_2$ and $\ell \equiv r_1$, then the tuple $(u, r \rightarrow \ell, v)$ does not satisfy at least one of the conditions in [17, Definition 9.1].

We did not give here the 4th condition in detail as it would not be important to us. Suffice it to know that if one can find the set of all tuples $[u, r_1 \rightarrow r_2, v]$ which satisfy conditions (1)-(3) then one can check for each one of them if it satisfies condition (4) and thus, if it belongs to B or not. Given the set B , the algorithm in [17], shows how to associate an element of $\text{DG}(\mathcal{K}', p)$ with each tuple in B . The set of elements associated with tuples in B is a generating set of $\text{DG}(\mathcal{K}', p)$. If \mathcal{P} is complete then the generating set is minimal.

The process of finding the element of $\text{DG}(\mathcal{K}', p)$ associated with a given tuple in B is straightforward. The difficult parts in the algorithm are (1) finding a completion \mathcal{P}' and (2) finding the set of tuples B . We note that if \mathcal{K}' is a finite directed 2-complex, then \mathcal{P} is finite. However, it does not imply the existence of a finite completion. Even if there is a finite completion \mathcal{P}' , the set B might still be infinite and there is no simple way to find it. Thus, implementing the algorithm from [17, Lemma 9.11] is often impractical.

If the diagram group in question is the closure of a subgroup H of F , the situation is ameliorated. While the first problem remains valid, the task of constructing the set B becomes very simple. Let H be a subgroup of F . Let $\mathcal{L} = \mathcal{L}(H)$ and let $p = p_{\mathcal{L}(H)}$ be the distinguished edge of $\mathcal{L}(H)$. Let \mathcal{P} be the semigroup presentation associated with \mathcal{L} and let S be the semigroup it defines. We note that every relation $r_1 \rightarrow r_2$ in \mathcal{P} is such that $|r_1| > |r_2|$ (indeed, the top path of a negative cell is longer than the bottom path of the cell). Thus, regardless of the order one fixes on the set of edges E , the presentation \mathcal{P} is as required. The simplification in the algorithm for finding a generating set of $\text{Cl}(H) \cong \text{DG}(\mathcal{L}, p)$ has two main reasons.

The first reason is that every diagram Δ over \mathcal{L} (or over the directed 2-complex \mathcal{L} with all vertices identified) can be viewed as a diagram over the Dunce hat \mathcal{K} . Moreover, as in the proof of Lemma 3.8, if Δ is reduced as a diagram over \mathcal{L} , then it is also reduced as a diagram over \mathcal{K} . Thus, if Δ is a reduced diagram over \mathcal{L} , then there is a horizontal 1-path in Δ , which passes through all the vertices of Δ and separates it to a concatenation of a positive subdiagram Δ^+ and a negative subdiagram Δ^- . We note that the implication for

the semigroup S is that if w_1 and w_2 are words over E , equal as elements of S , then to get from w_1 to w_2 one can apply a positive derivation over \mathcal{P} followed by a negative derivation (for terminology, see [25]).

The second reason is that as usual, if Δ is a diagram in the core, one can consider paths on Δ which can be mapped to paths on the core $\mathcal{L}(H)$ and use results from previous sections.

In the proof of the following proposition we would often consider diagrams Δ alternately as diagrams over \mathcal{L} and as diagrams over \mathcal{K} . When we refer to the label $\text{lab}(e)$ of an edge e or $\text{lab}(q)$ of a 1-path q in Δ , the label refers to the label of the edge or 1-path when Δ is viewed as a diagram over \mathcal{L} . In particular, $\text{lab}(e)$ is an edge of \mathcal{L} .

Let w_1, w_2 be words over E . We say that w_1 is a *left divisor* of w_2 in S if there is a word a over E such that $w_1 a$ is equal to w_2 in S . Right divisors are defined in a similar way.

Proposition 10.1. Let H be a subgroup of F . Let $\mathcal{L} = \mathcal{L}(H)$ and let $p = p_{\mathcal{L}(H)}$ be the distinguished edge of \mathcal{L} . Let \mathcal{P} be the semigroup presentation associated with \mathcal{L} and let S be the semigroup given by \mathcal{P} . Let w, w_1 and w_2 be words over the alphabet E . Then the following assertions hold.

- (1) w is a left divisor of p in S if and only if w is a 1-path in \mathcal{L} such that $\iota(w) = \iota(\mathcal{L})$.
- (2) Assume that w_1, w_2 are non-empty words over E such that w_1 and w_2 are left divisors of p in S . Then w_1 is equivalent to w_2 in S if and only if the 1-paths w_1 and w_2 satisfy $\tau(w_1) = \tau(w_2)$.

Proof. (1) Assume that w is a left divisor of p . By definition, there is a word a such that $wa = p$ in S . Since p is a 1-path in \mathcal{L} , wa is also a 1-path in \mathcal{L} and $\iota(wa) = \iota(p) = \iota(\mathcal{L})$, $\tau(wa) = \tau(\mathcal{L})$. In particular, w is a 1-path with initial vertex $\iota(\mathcal{L})$.

In the other direction, let w be a 1-path in \mathcal{L} with initial vertex $\iota(\mathcal{L})$. Assume by contradiction that w is not a left divisor of p . We can assume that w is a minimal 1-path with these properties. Clearly, w is not a trivial path (i.e., there are edges in the path w). Similarly, w is not composed of a single edge. Otherwise, let u be a path on \mathcal{L} such that $u^+ = w$ (where w is viewed as an edge). $u \equiv 0^k$ for some $k \geq 0$, since w is a left boundary edge of \mathcal{L} . Let Ψ be the minimal tree-diagram over \mathcal{K} with branch u . Since u labels a path on \mathcal{L} , Ψ can be viewed as a diagram over \mathcal{L} with $\text{lab}(\mathbf{top}(\Psi)) \equiv p$ and $\text{lab}(\mathbf{bot}(\Psi)) \equiv wd$ for some word d over E . Then $wd = p$ in S and w is a left divisor of p , which contradicts the assumption.

Thus, we can write $w \equiv w'c$ where c is the last letter of w and w' is not empty. The minimality of w implies that w' is a left divisor of p . Thus, there is a reduced $(p, w'q)$ -diagram Δ over \mathcal{L} for some word q over E . We let e be the $|w'| + 1$ edge on $\mathbf{bot}(\Delta)$, so that $\text{lab}(e)$ is the first letter of q . We let e' be the edge on the horizontal 1-path of Δ such that $e'_- = e_-$ and let $c' \equiv \text{lab}(e')$. Then c' is an edge of \mathcal{L} such that $c'_- = \tau(w') = c_-$. If Δ is viewed as a diagram over \mathcal{K} , then the positive subdiagram Δ^+ is a tree-diagram. Let u be the path on Δ^+ with terminal edge e' . Let Ψ be the minimal tree-diagram over \mathcal{K} with branch u . Then Ψ can be viewed as a subdiagram of Δ with $\mathbf{top}(\Psi) = \mathbf{top}(\Delta^+)$ (see Figure 10.11). The edge e' lies on the bottom path of Ψ . Thus, $\mathbf{bot}(\Psi) = ae'b$, where a and b are the suitable 1-paths in Δ . Let $a' \equiv \text{lab}(a)$ and $b' \equiv \text{lab}(b)$. It follows that Δ has an (a', w') -subdiagram. Hence a' and w' represent the same element of S . We claim that $a'c$ is a left divisor of p

in S . That would yield the result, as $w \equiv w'c$ is equivalent to $a'c$ in S . We note that, as 1-paths on \mathcal{L} , $\tau(a') = \tau(w')$. Thus, $a'c$ is a 1-path in \mathcal{L} .

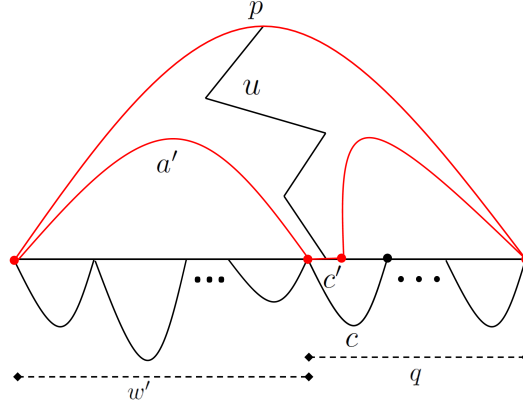


Figure 10.10: The diagram Δ . The top and bottom paths of the subdiagram Ψ are colored red. The labels of edges or 1-paths in the figure are their labels when Δ is viewed as a diagram over \mathcal{L} .

Since Δ is a diagram over \mathcal{L} with $\text{lab}(\text{top}(\Delta)) \equiv p$, the path u in Δ implies that u labels a path on \mathcal{L} such that $u^+ = c'$. Since $c_- = c'_- \neq \iota(\mathcal{L})$, by Lemma 6.6, c and c' belong to the same connected component of $\Gamma(H)$. Let v be a path on \mathcal{L} with terminal edge c . By Remark 6.4, there are $m, n \geq 0$ such that $u0^m$ and $v0^n$ label paths on \mathcal{L} and such that $(u0^m)^+ = (v0^n)^+$ on \mathcal{L} . Thus, by Lemma 6.1, there is a diagram Δ' in F , accepted by \mathcal{L} , with a pair of branches $u0^m \rightarrow v0^n$ (see Figure 10.11). Let d be the edge on the horizontal 1-path of Δ' such that the positive branch $u0^m$ and the negative branch $v0^n$ terminate on d . We let d_1 be the terminal edge of the positive path u on Δ' and d_2 be the terminal edge of the negative branch v on Δ' . Then $d_- = d_{1-} = d_{2-}$.

Consider Δ' as a diagram over \mathcal{L} . Then $\text{lab}(d_1) \equiv c'$ and $\text{lab}(d_2) \equiv c$. Since u labels a positive path on Δ' , the tree-diagram Ψ can be viewed as a subdiagram of Δ'^+ with $\text{top}(\Psi) = \text{top}(\Delta')$. In particular, the bottom path of Ψ is a 1-path in Δ' . Recall that $\text{lab}(\text{bot}(\Psi)) \equiv a'c'b'$ where c' is the label of the edge u^+ of Ψ , i.e., the edge d_1 of Δ' . Since $d_{1-} = d_{2-}$ and $\text{lab}(d_2) \equiv c$, we get that $a'c$ labels a 1-path in Δ' , which starts from $\iota(\Delta')$. Extending the 1-path, we get a 1-path with label $a'cs$ (for some word s) with initial vertex $\iota(\Delta')$ and terminal vertex $\tau(\Delta')$. Then Δ' has a $(p, a'cs)$ -subdiagram, which implies that $a'c$ is a left divisor of p in S .

(2) Let $w_1, w_2 \neq \emptyset$ be left divisors of p in S . By part (1), w_1 and w_2 are 1-paths in \mathcal{L} with initial vertex $\iota(\mathcal{L})$. If w_1 and w_2 are equivalent in S , then there is a (w_1, w_2) -diagram over \mathcal{L} . Then $\tau(w_1) = \tau(w_2)$ as required. In the other direction, assume that $\tau(w_1) \neq \tau(w_2)$. Let q_1 and q_2 be words over E such that $w_i q_i$ is equal to p in S . If q_1 is empty, then $w_1 = p$ in S and so, $\tau(w_1) = \tau(p) = \tau(\mathcal{L})$. It follows that $\tau(w_2) = \tau(\mathcal{L})$. As $\tau(\mathcal{L})$ has no outgoing edges, q_2 is also empty. It follows that $w_2 = p$ in S . Thus, $w_1 = w_2$ in S .

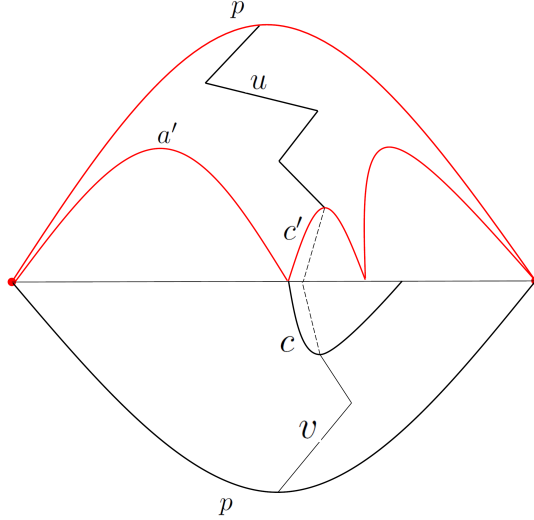


Figure 10.11: The diagram Δ' . The top and bottom paths of the subdiagram Ψ are colored red. The labels of edges or 1-paths in the figure are their labels when Δ' is viewed as a diagram over \mathcal{L} .

Therefore, we can assume that q_1 and q_2 are not empty words. Let Δ_i , $i = 1, 2$ be a $(p, w_i q_i)$ -diagram over \mathcal{L} . Let c_i be the first letter of the word q_i . As $\tau(w_1) = \tau(w_2)$, the edges c_1 and c_2 of \mathcal{L} have the same initial vertex. We apply an argument similar to the one in part (1). Let e_i be the $|w_i| + 1$ edge on $\mathbf{bot}(\Delta_i)$, so that $\text{lab}(e_i) \equiv c_i$. Let e'_i be the edge on the horizontal 1-path of Δ_i with the same initial vertex as e_i . Let $c'_i \equiv \text{lab}(e'_i)$ and let u_i be the path on Δ_i^+ with terminal edge e'_i . Let Ψ_i be the minimal tree-diagram with path u_i . Ψ_i can be viewed as a subdiagram of Δ_i^+ with $\mathbf{top}(\Psi_i) = \mathbf{top}(\Delta_i)$ and $\mathbf{bot}(\Psi_i) = a_i e'_i b_i$. We let $a'_i \equiv \text{lab}(a_i)$ and $b'_i \equiv \text{lab}(b_i)$. Then Δ_i has an (a'_i, w_i) -subdiagram and so, a'_i is equivalent to w_i in S .

Since $c'_{i-} = \tau(w_i)$, we have $c'_{1-} = c'_{2-}$ in \mathcal{L} . We note that u_i labels a path on \mathcal{L} with terminal edge c'_i . Thus, by Proposition 6.6 and Remark 6.4 there are $m_i \geq 0$ such that $u_i 0^{m_i}$ labels a path on \mathcal{L} and such that $(u_1 0^{m_1})^+ = (u_2 0^{m_2})^+$ in \mathcal{L} . By Lemma 6.1, there is a diagram Δ' accepted by \mathcal{L} with a pair of branches $u_1 0^{m_1} \rightarrow u_2 0^{m_2}$. The minimality of Ψ_i implies that if Δ' is viewed as a diagram over \mathcal{L} , then Ψ_1 is a subdiagram of Δ'^+ with $\mathbf{top}(\Psi_1) = \mathbf{top}(\Delta'^+)$ and Ψ_2^{-1} is a subdiagram of Δ'^- with $\mathbf{bot}(\Psi_2^{-1}) = \mathbf{bot}(\Delta'^-)$. Then it is easy to see (Figure 10.12) that Δ' has an (a'_1, a'_2) -subdiagram. Hence, a'_1 and a'_2 are equivalent in S . Since a'_i is equivalent to w_i in S , $w_1 = w_2$ in S , as required. \square

In a similar way, one can prove the following right-left analogue of Proposition 10.1.

Proposition 10.2. Let H be a subgroup of F . Let $\mathcal{L} = \mathcal{L}(H)$ and let $p = p_{\mathcal{L}(H)}$ be the

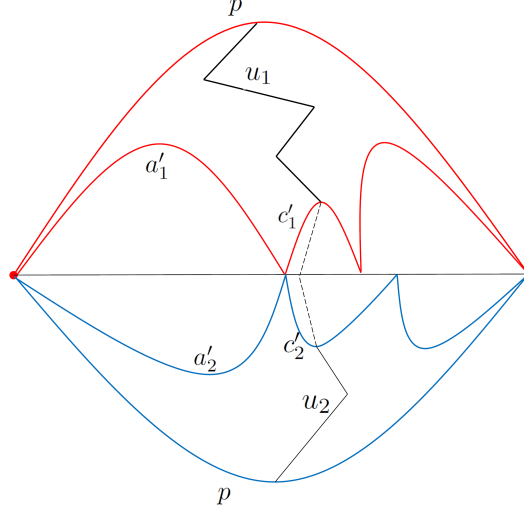


Figure 10.12: The diagram Δ' . The top and bottom paths of Ψ_1 and of Ψ_2^{-1} are colored red and blue respectively. The labels of edges or 1-paths in the figure are their labels when Δ' is viewed as a diagram over \mathcal{L} .

distinguished edge of \mathcal{L} . Let \mathcal{P} be the semigroup presentation associated with \mathcal{L} and let S be the semigroup given by \mathcal{P} . Let w, w_1 and w_2 be words over the alphabet E . Then the following assertions hold.

- (1) w is a right divisor of p in S if and only if w is a 1-path in \mathcal{L} such that $\tau(w) = \tau(\mathcal{L})$.
- (2) Assume that w_1, w_2 are non-empty words over E such that w_1 and w_2 are right divisors of p in S . Then w_1 is equivalent to w_2 in S if and only if the 1-paths w_1 and w_2 satisfy $\iota(w_1) = \iota(w_2)$.

Corollary 10.3. Let H be a subgroup of F and let $p = p_{\mathcal{L}(H)}$ be the distinguished edge of $\mathcal{L}(H)$. Let \mathcal{P} be the semigroup presentation associated with $\mathcal{L} = \mathcal{L}(H)$ and let S be the semigroup given by \mathcal{P} . Let \mathcal{P}' be a completion of \mathcal{P} with the properties described at the top of the section. Then for each relation $r_1 \rightarrow r_2$ in \mathcal{P}' there is at most one tuple $[u, r_1 \rightarrow r_2, v]$ which satisfies conditions (1)-(3) out of conditions (1)-(4) listed at the top of the section.

Proof. Let $r_1 \rightarrow r_2$ be a relation in \mathcal{P}' . If r_1 is not a 1-path in \mathcal{L} , then there is no tuple $[u, r_1 \rightarrow r_2, v]$ as described. Indeed, condition (3) says that $p = ur_1v$ in S . Since p is a 1-path in \mathcal{L} , that would imply that ur_1v , and as such, that r_1 , is a 1-path in \mathcal{L} . Assume that r_1 is a 1-path in \mathcal{L} . If $[u, r_1 \rightarrow r_2, v]$ is as described, then u and v must be 1-paths in \mathcal{L} such that $\iota(u) = \iota(\mathcal{L})$, $\tau(u) = \iota(r_1)$, $\iota(v) = \tau(r_1)$ and $\tau(v) = \tau(\mathcal{L})$. The requirement that u and v are reduced implies, By Propositions 10.1(2) and 10.2(2), that there is only one choice for u and v ; namely, u is the unique 1-path in \mathcal{L} with $\iota(u) = \iota(\mathcal{L})$ and $\tau(u) = \iota(r_1)$ such that the word u is reduced over \mathcal{P}' . Similarly, there is only one option for the choice of v . Finally,

we note that if u and v are taken to be 1-paths as described, then conditions (1) and (2) are satisfied. Since ur_1v is a 1-path in \mathcal{L} such that $\iota(ur_1v) = \iota(\mathcal{L})$ and $\tau(ur_1v) = \tau(\mathcal{L})$, by Proposition 10.1(2), ur_1v is equivalent to p in S . Hence, condition (3) holds for this choice as well. \square

The proof of Corollary 10.3 shows that given a subgroup $H \leq F$ with finite $\mathcal{L}(H)$ and a completion \mathcal{P}' of the semigroup presentation \mathcal{P} associated with $\mathcal{L}(H)$, there is a simple algorithm for finding the set of tuples B (and thus, for implementing the algorithm from [17] for finding a generating set of $\text{Cl}(H)$). Indeed, for each inner vertex x of $\mathcal{L}(H)$ one can find 1-paths p and q on $\mathcal{L}(H)$ such that $\iota(p) = \iota(\mathcal{L}(H))$, $\tau(p) = x$, $\iota(q) = x$ and $\tau(q) = \tau(\mathcal{L}(H))$. Then applying rewriting rules from \mathcal{P}' one can find such 1-paths p and q , which are reduced as words in the alphabet E over \mathcal{P}' .

10.2 Core 2-automata

In previous sections we sometimes described the core $\mathcal{L}(H)$ of a subgroup $H \leq F$ using a labeled binary tree. In this section we generalize this notion and make it precise. All 2-automata \mathcal{L} considered in this section are 2-automata over the Duncie hat \mathcal{K} with distinguished 1-paths $p_{\mathcal{L}} = q_{\mathcal{L}}$ composed of one edge of \mathcal{L} . We always assume that the immersion $\psi_{\mathcal{L}}$ from \mathcal{L} into \mathcal{K} maps every positive cell of \mathcal{L} to the positive cell of \mathcal{K} . Recall that \mathcal{L} is a folded-automaton if no foldings are applicable to it. The proof of [15, Lemma 3.21] shows that if \mathcal{L} is a folded-automaton and Δ is a diagram in F accepted by \mathcal{L} then the reduced diagram equivalent to Δ is also accepted by \mathcal{L} . Thus, one can talk about the subgroup of F accepted by \mathcal{L} .

Lemma 10.4. Let H be a subgroup of F . Then H is a closed subgroup of F if and only if there is a folded-automaton \mathcal{L} over the Duncie hat \mathcal{K} such that H is the subgroup of F accepted by \mathcal{L} .

Proof. If H is a closed subgroup of F then one can take $\mathcal{L} = \mathcal{L}(H)$. In the other direction, if H is the subgroup of all diagrams in F accepted by \mathcal{L} then Remark 3.10 implies that H is closed for components. Then by Corollary 5.7, H is a closed subgroup of F . \square

Let \mathcal{L} be a folded automaton over \mathcal{K} . The subgroup of F accepted by \mathcal{L} is naturally isomorphic to the diagram group $G = \text{DG}(\mathcal{L}, p_{\mathcal{L}})$ (indeed, the proof is identical to that of Lemma 3.8). In Section 10.1 we have seen that if \mathcal{L} is the core $\mathcal{L}(H)$ of some subgroup H of F then the algorithm from [17] for finding a generating set of G can be simplified. For this and for other reasons (see Section 10.3 below), it is useful to determine if \mathcal{L} coincides with $\mathcal{L}(H)$ for some subgroup $H \leq F$. We are only interested in \mathcal{L} coinciding with $\mathcal{L}(H)$ when all vertices of \mathcal{L} and all vertices of $\mathcal{L}(H)$ are identified to a single vertex. Indeed, by Remark 2.3, the diagram groups G and $\text{DG}(\mathcal{L}(H), p_{\mathcal{L}(H)}) \cong \text{Cl}(H)$ are not affected by identification of vertices in \mathcal{L} and in $\mathcal{L}(H)$.

Definition 10.5. Let \mathcal{L} be a folded-automaton over \mathcal{K} . We say that \mathcal{L} is a *core-automaton* if there is a subgroup $H \leq F$ and a bijective morphism ϕ from $\mathcal{L}(H)$ with all vertices identified to \mathcal{L} with all vertices identified.

We note that the naive approach for deciding if a folded automaton \mathcal{L} is a core automaton is to find a generating set X of $\text{DG}(\mathcal{L}, p_{\mathcal{L}})$, let H be the subgroup of F generated by X (where elements in X are viewed as reduced diagrams in F) and construct the core $\mathcal{L}(H)$. Then one has to check if \mathcal{L} and $\mathcal{L}(H)$ coincide up to identification of vertices. This approach would sometimes work, but as we are trying to simplify the process of finding a generating set for $\text{DG}(\mathcal{L}, p_{\mathcal{L}})$ we consider a different approach. Namely, we associate labeled binary trees $T_{\mathcal{L}}$ and $T_{\mathcal{L}}^{\min}$ with the folded-automaton \mathcal{L} . The folded automaton will be a core automaton if and only if the tree $T_{\mathcal{L}}^{\min}$ satisfies certain properties (see Lemma 10.9 below).

Given a labeled binary tree T , a *path* p in T is always a simple path starting from the root. Every path is labeled by a finite binary word u . As for paths on diagrams, we will rarely distinguish between the path p and its label u . Similarly, we will denote by p^+ or u^+ the terminal vertex of the path p in T . $\text{lab}(u^+)$ or $\text{lab}(v^+)$ will denote the label of this terminal vertex. An *inner* vertex of T is a vertex which is not a leaf. *Brother vertices* of T are distinct vertices with a common father.

Let \mathcal{L} be a folded-automaton. The labeled binary tree $T_{\mathcal{L}}$ associated with \mathcal{L} is defined as follows. The labels of vertices in $T_{\mathcal{L}}$ are edges of \mathcal{L} . Recall (Section 4), that each finite binary word u labels at most one path on \mathcal{L} . We let $T_{\mathcal{L}}$ be the maximal binary tree such that for every path u in $T_{\mathcal{L}}$, the finite binary word u labels a path on \mathcal{L} . For example, if every edge in \mathcal{L} is the top edge of some cell, then $T_{\mathcal{L}}$ is the complete infinite binary tree. The label of each vertex u^+ of $T_{\mathcal{L}}$ is the edge u^+ of \mathcal{L} .

Notice that every caret in $T_{\mathcal{L}}$ is labeled with accordance with the top and bottom edges of some positive cell in \mathcal{L} . In fact, $T_{\mathcal{L}}$ can be constructed inductively as follows. One starts with a root labeled by the distinguished edge $p_{\mathcal{L}}$ of \mathcal{L} . Whenever there is a leaf in the tree whose label is the top edge of some positive cell π in \mathcal{L} , one attaches a caret to the leaf and labels the left (resp. right) leaf of the caret by the left (resp. right) bottom edge of π .

Now let T be a rooted subtree of $T_{\mathcal{L}}$, maximal with respect to the property that there is no pair of distinct inner vertices in T which have the same label. If ℓ is a leaf of T and ℓ does not share a label with any inner vertex in T , then ℓ must be a leaf of $T_{\mathcal{L}}$. Indeed, otherwise one could attach the caret of $T_{\mathcal{L}}$ with root ℓ to the subtree T and get a larger subtree where no pair of distinct inner vertices share a label.

If the leaf ℓ shares a label with some inner vertex x of T , then in $T_{\mathcal{L}}$, ℓ has two children. Each child of ℓ is labeled as the respective child of x . Continuing in this manner, one can show that it is possible to get $T_{\mathcal{L}}$ from T , by inductively attaching carets to leaves which share their label with inner vertices of T and labeling the new leaves appropriately. It follows that T and $T_{\mathcal{L}}$ have the same set of labeled carets. Since no labeled caret appears in T more than once, T is a minimal subtree of $T_{\mathcal{L}}$ with respect to the property that the sets of labeled carets of T and $T_{\mathcal{L}}$ coincide.

We let a *minimal tree associated with \mathcal{L}* , denoted $T_{\mathcal{L}}^{\min}$ be a tree T as described in the preceding paragraph. We note that a minimal tree associated with \mathcal{L} is not unique. However, the label of the root of $T_{\mathcal{L}}^{\min}$ and the set of labeled carets of $T_{\mathcal{L}}^{\min}$ are determined uniquely by \mathcal{L} . Thus, we can consider different minimal trees associated with \mathcal{L} to be equivalent. All labeled binary trees which appeared in this paper so far were minimal trees associated with the cores $\mathcal{L}(H)$ they described.

Lemma 10.6. Let \mathcal{L} be a folded-automaton over \mathcal{K} . Let $T_{\mathcal{L}}^{\min}$ be an associated minimal

tree. Assume that for all u and v which label paths on $T_{\mathcal{L}}^{\min}$ such that u^+ and v^+ share a label, there is a diagram $\Delta_{u,v}$ accepted by \mathcal{L} with a pair of branches $u \rightarrow v$. Let K be the subgroup of F generated by the diagrams $\Delta_{u,v}$ for each pair of paths u, v on $T_{\mathcal{L}}^{\min}$ with $\text{lab}(u^+) \equiv \text{lab}(v^+)$. Then $\text{Cl}(K)$ is the subgroup of F accepted by \mathcal{L} . In addition, there is an injective morphism ϕ from the core $\mathcal{L}(K)$ with all vertices identified to \mathcal{L} with all vertices identified.

Proof. Since \mathcal{L} is a folded-automaton and all diagrams $\Delta_{u,v}$ are accepted by \mathcal{L} , all diagrams in K are accepted by \mathcal{L} . Thus, by Remark 3.10 and Corollary 5.7, $\text{Cl}(K)$ is accepted by \mathcal{L} . To prove the opposite inclusion we need the following lemma.

Lemma 10.7. Let u and v be paths on the tree $T_{\mathcal{L}}$ associated with \mathcal{L} such that $\text{lab}(u^+) \equiv \text{lab}(v^+)$. Then there is an element $k \in K$ with a pair of branches $u \rightarrow v$.

Proof. The proof is similar to the proof of Lemma 4.6 for foldings of type 1. The diagrams $\Delta_{u,v}$ show that the lemma holds if the paths u and v belong to the subtree $T_{\mathcal{L}}^{\min}$ of $T_{\mathcal{L}}$. Recall that $T_{\mathcal{L}}$ can be constructed from $T_{\mathcal{L}}^{\min}$ by inductively attaching caret. We claim that whenever a caret is attached during the inductive construction, the lemma remains true for the constructed tree.

Let T be a subtree of $T_{\mathcal{L}}$ resulting from $T_{\mathcal{L}}^{\min}$ by the attachment of finitely many labeled carets and assume that the lemma holds for T . Let w_1^+ be a leaf of T which shares a label with some inner vertex w_2^+ of $T_{\mathcal{L}}^{\min}$ (for some finite binary words w_1 and w_2). We let T' be the subtree of $T_{\mathcal{L}}$ which results from T if one attaches a caret to w_1^+ and labels the left (resp. right) child of w_1^+ by the label of the left (resp. right) child of w_2^+ .

Let u and v be paths on T' such that $\text{lab}(u^+) \equiv \text{lab}(v^+)$. If w_1 is not a proper prefix of u nor a proper prefix of v , then u^+ and v^+ are vertices of the subtree T and the lemma holds by the induction hypothesis. Thus, we can assume that w_1 is a proper prefix of u . It follows that $u \equiv w_1 0$ or $u \equiv w_1 1$. We consider two possible cases.

(1) w_1 is not a proper prefix of v ; i.e. v^+ is a vertex of T .

We assume without loss of generality that $u \equiv w_1 0$, the other case being similar. Since $\text{lab}(v^+) \equiv \text{lab}(u^+) \equiv \text{lab}((w_1 0)^+) \equiv \text{lab}((w_2 0)^+)$ and v^+ and $(w_2 0)^+$ are vertices of T (indeed, w_2^+ was an inner vertex of T), by the induction hypothesis, there is an element $k_1 \in K$ with a pair of branches $w_2 0 \rightarrow v$. Similarly, since $\text{lab}(w_1^+) \equiv \text{lab}(w_2^+)$ and w_1^+, w_2^+ are vertices of T , by the induction hypothesis, there is an element $k_2 \in K$ with a pair of branches $w_1 \rightarrow w_2$. Thus, $u \equiv w_1 0 \rightarrow w_2 0$ is a pair of branches of k_2 . It follows that $u \rightarrow v$ is a pair of branches of $k_2 k_1$, as required.

(2) w_1 is a proper prefix of v .

Then $\text{lab}(v^+) \equiv \text{lab}(u^+) \equiv \text{lab}((w_1 0)^+) \equiv \text{lab}((w_2 0)^+)$. Since $(w_2 0)^+$ is a vertex of T , we have by the previous case that there are elements $k_1, k_2 \in K$ with pairs of branches $u \rightarrow w_2 0$ and $v \rightarrow w_2 0$, respectively. Then $k_1 k_2^{-1}$ has a pair of branches $u \rightarrow v$ as required. \square

By Lemma 10.7 and the definition of $T_{\mathcal{L}}$, we have that for every pair of paths u and v on \mathcal{L} such that $u^+ = v^+$, there is an element $k \in K$ with a pair of branches $u \rightarrow v$. Now, let Δ be a reduced diagram accepted by \mathcal{L} and let $u_i \rightarrow v_i$, $i = 1, \dots, n$ be the pairs of branches of Δ . Then for every $i \in \{1, \dots, n\}$, u_i and v_i label paths on \mathcal{L} such that $u_i^+ = v_i^+$. Therefore, for each i , there is an element k_i in K such that Δ and k_i coincide on $[u_i]$. Therefore, Δ is

dyadic-piecewise- K and by Lemma 5.5, Δ belongs to $\text{Cl}(K)$. Thus, $\text{Cl}(K)$ is the subgroup of F accepted by \mathcal{L} .

For the rest of the proof we assume that all vertices of $\mathcal{L}(K)$ and all vertices of \mathcal{L} are identified to a single vertex. To define the injective morphism ϕ from $\mathcal{L}(K)$ to \mathcal{L} we only consider edges and cells of the 2-automata (the unique vertex of $\mathcal{L}(K)$ is mapped to the unique vertex of \mathcal{L}). Hence, the non-standard notation should not cause confusion. To prove that there is an injective morphism ϕ from $\mathcal{L}(K)$ to \mathcal{L} , we let $X = \{\Delta_i : i \in \mathcal{I}\}$ be a set of reduced diagrams generating K . Since every diagram Δ_i is accepted by $\mathcal{L}(K)$, there is a natural morphism ψ_{Δ_i} from Δ_i to $\mathcal{L}(K)$. Moreover, the construction of $\mathcal{L}(K)$ (starting from the bouquet of spheres \mathcal{L}') shows that every edge (resp. cell) of $\mathcal{L}(K)$ is the image under ψ_{Δ_i} of an edge (resp. cell) of Δ_i , for some $i \in \mathcal{I}$. Since each diagram Δ_i is accepted by \mathcal{L} , there is a morphism ψ'_{Δ_i} from Δ_i to \mathcal{L} .

Let e be an edge of $\mathcal{L}(K)$. We choose $i \in \mathcal{I}$ and an edge e' in Δ_i such that $\psi_{\Delta_i}(e') = e$ and let $\phi(e) = \psi'_{\Delta_i}(e')$. We define the action of ϕ on cells of $\mathcal{L}(K)$ in a similar way. It is easy to see that if ϕ is well defined (i.e., does not depend on the choice of preimages of edges and cells of $\mathcal{L}(K)$ in the diagrams Δ_i , $i \in \mathcal{I}$), then ϕ is a morphism from $\mathcal{L}(K)$ to \mathcal{L} . Indeed, the definition of ϕ respects adjacency of edges and cells in $\mathcal{L}(K)$.

We show that the action of ϕ on edges of $\mathcal{L}(K)$ is well defined. It will follow that the action on cells is well defined as well. Let e be an edge of $\mathcal{L}(K)$ and let $i, j \in \mathcal{I}$ and e_1, e_2 be edges of Δ_i and Δ_j respectively such that $\psi_{\Delta_i}(e_1) = e$ and $\psi_{\Delta_j}(e_2) = e$. We claim that $\psi'_{\Delta_i}(e_1) = \psi'_{\Delta_j}(e_2)$.

Let u be a (positive or negative) path to e_1 on the diagram Δ_i and let v be a (positive or negative) path to e_2 on the diagram Δ_j . Since Δ_i and Δ_j are accepted by \mathcal{L} , u and v label paths on \mathcal{L} , such that $u^+ = \psi'_{\Delta_i}(e_1)$ and $v^+ = \psi'_{\Delta_j}(e_2)$ in \mathcal{L} . Similarly, u and v label paths on $\mathcal{L}(K)$ such that $u^+ = \psi_{\Delta_i}(e_1) = e = \psi_{\Delta_j}(e_2) = v^+$. Thus, by Lemma 6.1, there is a diagram Δ in $\text{Cl}(K)$ with a pair of branches $u \rightarrow v$. If Δ is reduced, then it is accepted by \mathcal{L} and so $u^+ = v^+$ on \mathcal{L} . Otherwise, there are subpaths u_1 and v_1 such that $u \equiv u_1 s$, $v \equiv v_1 s$ and $u_1 \rightarrow v_1$ is a pair of branches of the reduced diagram equivalent to Δ . It follows that $u_1^+ = v_1^+$ on \mathcal{L} , which implies that $u^+ = v^+$ on \mathcal{L} . Thus, $\psi'_{\Delta_i}(e_1) = \psi'_{\Delta_j}(e_2)$ as required. Hence, ϕ is well defined on edges of $\mathcal{L}(K)$.

Now, let π be a positive cell of $\mathcal{L}(K)$. Let π_1 and π_2 be cells of diagrams Δ_i and Δ_j for $i, j \in \mathcal{I}$ such that $\psi_{\Delta_i}(\pi_1) = \pi$ and $\psi_{\Delta_j}(\pi_2) = \pi$. Then $\psi'_{\Delta_i}(\text{top}(\pi_1)) = \phi(\text{top}(\pi)) = \psi'_{\Delta_j}(\text{top}(\pi_2))$ and $\psi'_{\Delta_i}(\text{bot}(\pi_1)) = \phi(\text{bot}(\pi)) = \psi'_{\Delta_j}(\text{bot}(\pi_2))$. As the top and bottom paths of a cell in \mathcal{L} determine the cell uniquely, $\psi'_{\Delta_i}(\pi_1) = \psi'_{\Delta_j}(\pi_2)$.

It remains to prove that ϕ is injective on edges and cells. We prove injectivity on edges of $\mathcal{L}(K)$. As before, that would imply that ϕ is also injective on cells. Let e_1 and e_2 be two edges of $\mathcal{L}(K)$ such that $\phi(e_1) = \phi(e_2)$. Let e'_1 and e'_2 be edges of diagrams Δ_i and Δ_j , $i, j \in \mathcal{I}$ such that $\psi_{\Delta_i}(e'_1) = e_1$ and $\psi_{\Delta_j}(e'_2) = e_2$. Let u and v be paths on Δ_i and Δ_j respectively such that $u^+ = e'_1$ and $v^+ = e'_2$. Then u and v label paths on \mathcal{L} such that $u^+ = \phi(e_1) = \phi(e_2) = v^+$. Thus, by Lemma 10.7 there is an element $k \in K$ with a pair of branches $u \rightarrow v$. By Lemma 4.8, that implies that on $\mathcal{L}(K)$ we also have $e_1 = u^+ = v^+ = e_2$. \square

Corollary 10.8. Let H be a subgroup of F and let $\mathcal{L}(H)$ be the core of H . If $\mathcal{L}(H)$ is a finite directed 2-complex then $\text{Cl}(H) = \text{Cl}(K)$ where K is a finitely generated subgroup of

F .

Proof. Since $\mathcal{L}(H)$ is finite, the tree $T_{\mathcal{L}(H)}^{\min}$ is finite. By Lemma 6.1, for each pair of paths u and v such that u^+ and v^+ share a label in $T_{\mathcal{L}(H)}^{\min}$ (and thus, $u^+ = v^+$ on $\mathcal{L}(H)$), there is a diagram $\Delta_{u,v}$ accepted by $\mathcal{L}(H)$ with a pair of branches $u \rightarrow v$. Then by Lemma 10.6, the group K generated by the diagrams $\Delta_{u,v}$ satisfies $\text{Cl}(K) = \text{Cl}(H)$. \square

Lemma 10.9. Let \mathcal{L} be a folded-automaton over \mathcal{K} with minimal associated tree $T_{\mathcal{L}}^{\min}$. Then \mathcal{L} is a core-automaton if and only if the following conditions are satisfied.

- (1) \mathcal{L} is *given* by the tree $T_{\mathcal{L}}^{\min}$, i.e., for every edge (resp. cell) of \mathcal{L} there is a vertex (resp. caret) in $T_{\mathcal{L}}^{\min}$ labeled accordingly.
- (2) If x and y are brother leaves of $T_{\mathcal{L}}^{\min}$, then either x shares a label with some vertex $z \neq x$ of T or y shares a label with some vertex $z \neq y$ in T .
- (3) For each pair of paths u, v in $T_{\mathcal{L}}^{\min}$ such that u^+ and v^+ share a label, there is a diagram $\Delta_{u,v}$ such that $\Delta_{u,v}$ is accepted by \mathcal{L} and has a pair of branches $u \rightarrow v$.

Proof. If K is a subgroup of F then the minimal tree associated with $\mathcal{L}(K)$ satisfies conditions (1)-(3) from the lemma. Indeed, for condition (1) we note that for any edge e (resp. cell π) in $\mathcal{L}(K)$ there is a path u on $\mathcal{L}(K)$ terminating on the edge e (resp. “passing through” the cell π). Thus, there is a vertex (resp. caret) in $T_{\mathcal{L}(K)}$ and thus, in $T_{\mathcal{L}(K)}^{\min}$, labeled by the edge e (resp. in accordance with the top and bottom paths of π).

For condition (2), assume that x and y are brother leaves of $T_{\mathcal{L}(K)}^{\min}$ (such that x is to the left of y) and that each of them does not share a label with any other vertex in the tree. In particular, they do not share a label with any inner vertex of $T_{\mathcal{L}}^{\min}$ and as such, x and y are leaves of $T_{\mathcal{L}}$, when $T_{\mathcal{L}}^{\min}$ is viewed as a subtree of it. Let $e_1 \equiv \text{lab}(x)$ and $e_2 \equiv \text{lab}(y)$. Then in $\mathcal{L}(K)$, e_1 and e_2 form the bottom path of a cell π . We let $e = \text{top}(\pi)$ and note that e_1 and e_2 are not edges of any cell in $\mathcal{L}(K)$, other than π , and that e_1, e_2 and e are pairwise distinct. By the construction of $\mathcal{L}(K)$, there is a reduced diagram Δ in K with a pair of branches $u_1 \rightarrow v_1$ such that u_1 has an initial subpath u'_1 such that u'_1 labels a path on $\mathcal{L}(K)$ with $u'^+_1 = e_1$ (indeed, that is true for any edge of $\mathcal{L}(K)$). Since e_1 is not the top edge of any cell in π , we must have $u_1 \equiv u'_1$. In addition, u_1 must be of the form $u_1 \equiv u0$ where $u^+ = e$ in $\mathcal{L}(K)$. It follows that if $u_2 \rightarrow v_2$ is the following pair of branches, then $u_2 \equiv u10^k$ for some k . Since $(u1)^+ = e_2$ is not the top edge of any positive cell in $\mathcal{L}(K)$, $k = 0$. Thus, $u_1 \equiv u0$ and $u_2 \equiv u1$. Similar arguments show that $v^+_1 = u^+_1 = e_1$ and $v^+_2 = u^+_2 = e_2$ imply that $v_1 \equiv v0$ and $v_2 \equiv v1$ for some path v on $\mathcal{L}(K)$ with terminal edge e . Then, as in the proof of Lemma 7.7, the consecutive pairs of branches $u0 \rightarrow v0$ and $u1 \rightarrow v1$ imply that Δ is not reduced, in contradiction to the assumption. Hence condition (2) holds. Condition (3) holds for $\mathcal{L}(K)$ by Lemma 6.1.

In the other direction, assume that conditions (1)-(3) are satisfied. Let K be the subgroup of F generated by the diagrams $\Delta_{u,v}$ from condition (3). By Lemma 10.6, the subgroup of F accepted by \mathcal{L} is $\text{Cl}(K)$. Let ϕ be the morphism from $\mathcal{L}(K)$, with all vertices identified, to \mathcal{L} , with all vertices identified, constructed in the proof of Lemma 10.6. It suffices to prove that ϕ is surjective. Below, we do not distinguish between edges or cells of $\mathcal{L}(K)$ (resp. \mathcal{L})

and edges or cells of the 2-automaton with identified vertices. In particular, a path on $\mathcal{L}(K)$ (resp. \mathcal{L}) can be viewed as a path on the 2-automaton with identified vertices.

We observe that ϕ induces a mapping from paths on $\mathcal{L}(K)$ to paths on \mathcal{L} which preserves the labels of paths. To show that ϕ is surjective, it suffices to show that for any path u in \mathcal{L} , the word u labels a path on $\mathcal{L}(K)$. In fact, it suffices to consider paths u which correspond to paths on $T_{\mathcal{L}}^{\min}$. Indeed, by condition (1), for every edge e (resp. cell π) in \mathcal{L} , there is a path u in $T_{\mathcal{L}}^{\min}$ such that e is the last edge (resp. π is the last cell) visited by the path u on \mathcal{L} . Then, if u also labels a path on $\mathcal{L}(K)$ with $u^+ = e'$ (resp. π' being the last cell through which u passes) then $\phi(e') = e$ and $\phi(\pi') = \pi$.

Thus, let u be a path on $T_{\mathcal{L}}^{\min}$. We write $u \equiv va$ where $a \in \{0, 1\}$ is the last digit of u . Then v^+ is an inner vertex of $T_{\mathcal{L}}^{\min}$. As such, v can be prolonged to a path $w \equiv vs$, for a non empty suffix s such that w^+ is a leaf whose brother is also a leaf. It suffices to show that w labels a path on $\mathcal{L}(K)$. Indeed, since $s \neq \emptyset$, either $v0$ or $v1$ is an initial subpath of w , and as such labels a path on $\mathcal{L}(K)$. That implies that both $v0$ and $v1$ label paths on $\mathcal{L}(K)$ and as such, that $u \equiv va$ labels a path on $\mathcal{L}(K)$.

By condition (2) in the lemma, we can assume (by changing the last digit of w if necessary), that w^+ shares a label with some distinct vertex w'^+ of $T_{\mathcal{L}}^{\min}$. By condition (3) there is a diagram $\Delta = \Delta_{w,w'}$ accepted by \mathcal{L} with a pair of branches $w \rightarrow w'$. By the definition of K , $\Delta \in K$. Let Δ' be the reduced diagram equivalent to it. We claim that $w \rightarrow w'$ is a pair of branches of Δ' . Otherwise, there are prefixes w_1, w_2 and a non empty common suffix t such that $w \equiv w_1 t$, $w' \equiv w_2 t$ and $w_1 \rightarrow w_2$ is a pair of branches of Δ' . We note that w_1^+ and w_2^+ are distinct inner vertices of $T_{\mathcal{L}}^{\min}$, as t is not empty. Since Δ' is accepted by \mathcal{L} , it follows that the vertices $(w_1)^+$ and $(w_2)^+$ of $T_{\mathcal{L}}^{\min}$ share a label, in contradiction to the definition of $T_{\mathcal{L}}^{\min}$ as the maximal subtree of $T_{\mathcal{L}}$ where distinct inner vertices do not share their label. Thus, $w \rightarrow w'$ is a pair of branches of Δ' . Since Δ' is a reduced diagram in K it is accepted by $\mathcal{L}(K)$. In particular, the positive branch w labels a path on $\mathcal{L}(K)$ as required. \square

Let \mathcal{L} be a core-automaton and let u be a path on $T_{\mathcal{L}}^{\min}$. We denote by T_u the minimal labeled rooted subtree of $T_{\mathcal{L}}^{\min}$ with branch u . Assume that u is the k^{th} branch of T_u and that $\text{lab}(u^+) \equiv e$ in T_u . Then reading the labels of leaves of T_u from left to right yields a word $p_u e q_u$ in the alphabet E (where E is the set of edges of \mathcal{L}) where $|p_u| = k - 1$. The pair of words (p_u, q_u) is the *pair of words associated with the path u on $T_{\mathcal{L}}^{\min}$* .

Lemma 10.10. Let \mathcal{L} be a folded automaton and let $T_{\mathcal{L}}^{\min}$ be a minimal associated tree. Let \mathcal{P} be the semigroup presentation associated with the directed 2-complex \mathcal{L} (as in Section 10.1) and let S be the semigroup given by \mathcal{P} . Let u and v be a pair of paths in $T_{\mathcal{L}}^{\min}$ such that $\text{lab}(u^+) \equiv \text{lab}(v^+)$ and let (p_u, q_u) and (p_v, q_v) be the associated pairs of words. Then there is a diagram $\Delta_{u,v}$ accepted by \mathcal{L} with a pair of branches $u \rightarrow v$ if and only if $p_u = p_v$ and $q_u = q_v$ in the semigroup S .

Proof. Let e be the common label of u^+ and v^+ in $T_{\mathcal{L}}^{\min}$. We let Ψ_u and Ψ_v be the minimal tree-diagrams over \mathcal{K} with branches u and v respectively. Since u and v label paths on \mathcal{L} , Ψ_u and Ψ_v can be naturally viewed as diagrams over \mathcal{L} with $\text{lab}(\text{top}(\Psi_u)) \equiv \text{lab}(\text{top}(\Psi_v)) \equiv p_{\mathcal{L}}$. Clearly, $\text{lab}(\text{bot}(\Psi_u)) \equiv p_u e q_u$ and $\text{lab}(\text{bot}(\Psi_v)) \equiv p_v e q_v$.

Let $\Delta_{u,v}$ be a diagram in F accepted by \mathcal{L} with a pair of branches $u \rightarrow v$. Then Ψ_u can be viewed as a subdiagram of Δ^+ with $\text{top}(\Psi_u) = \text{top}(\Delta^+)$ and Ψ_v^{-1} can be viewed as

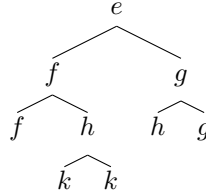
a subdiagram of Δ^- with $\mathbf{bot}(\Psi_v^{-1}) = \mathbf{bot}(\Delta^-)$. If one removes from Δ the subdiagrams Ψ_u (minus its bottom path) and Ψ_v^{-1} (minus its top path) as well as the terminal edge of the positive branch u and the negative branch v , one remains with 2 subdiagrams; a (p_u, p_v) -diagram and a (q_u, q_v) -diagram. Thus $p_u = p_v$ and $q_u = q_v$ in S .

In the opposite direction, assume that $p_u = p_v$ and $q_u = q_v$ in S . Then there is a (p_u, p_v) -diagram Δ_1 over \mathcal{L} and a (q_u, q_v) -diagram Δ_2 over \mathcal{L} (indeed, p_u, p_v, q_u, q_v are 1-paths on \mathcal{L}). Then $\psi_u \circ (\Delta_1 + \varepsilon(e) + \Delta_2) \circ \Psi_v^{-1}$ is a diagram in $\text{DG}(\mathcal{L}, p_{\mathcal{L}})$ with a pair of branches $u \rightarrow v$. \square

In general, Lemma 10.9 does not give an algorithm for deciding whether a folded-automaton is a core-automaton since we do not have an algorithm for deciding (in the notations of Lemma 10.10) whether two words w_1 and w_2 are equivalent in S . However, the condition is often useful.

Let T be a labeled binary tree such that (1) no two inner vertices of T share a label and (2) there are no distinct carets C_1 and C_2 in T such that the label of each leaf of C_1 coincides with the label of the respective leaf of C_2 . Then there is a folded automaton \mathcal{L} , given by a minimal tree $T_{\mathcal{L}}^{\min}$ such that $T_{\mathcal{L}}^{\min}$ coincides with T .

Example 10.11. Let \mathcal{L} be a folded-automaton given by the following minimal associated tree $T_{\mathcal{L}}^{\min}$.



Then \mathcal{L} is not a core-automaton.

Proof. We consider the paths $u \equiv 010$ and $v \equiv 011$ on $T_{\mathcal{L}}^{\min}$. They satisfy $\text{lab}(u^+) \equiv \text{lab}(v^+) \equiv k$. In the notations of Lemma 10.10, we have $(p_u, q_u) \equiv (f, kg)$ and $(p_v, q_v) \equiv (fk, g)$. Let $\mathcal{P} = \langle e, f, g, h, k \mid fg \rightarrow e, fh \rightarrow f, hg \rightarrow g, kk \rightarrow h \rangle$ be the semigroup presentation given by \mathcal{L} and let S be the semigroup with presentation \mathcal{P} . When \mathcal{P} is viewed as a rewriting system, it is confluent and terminating. Since no relation from \mathcal{P} is applicable to fk , nor to f , they are both reduced words over \mathcal{P} . Hence, they are not equivalent in S . Thus, by Lemma 10.9 and Lemma 10.10, \mathcal{L} is not a core-automaton. \square

10.3 Maximal subgroups of F of infinite index

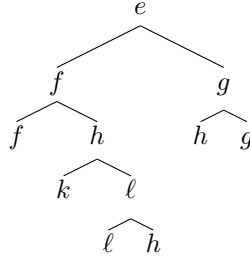
10.3.A Construction of maximal subgroups of F of infinite index which do not fix any number in $(0, 1)$

Recall that by [27, Proposition 1.4], for every $\alpha \in (0, 1)$, the stabilizer of α , $H_{\{\alpha\}}$ is a maximal subgroup of F . Savchuk asked [26, Problem 1.5] whether all maximal subgroups of F of infinite index are of this form. The core $\mathcal{L}(H)$ of a subgroup $H \leq F$ was applied in [15] to provide implicit examples of maximal subgroups of F of infinite index which do not fix any number in $(0, 1)$. Applying results from this paper, one can use the Stallings

2-core to provide explicit examples of such maximal subgroups. Indeed, in [15], we showed that $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$ is (1) a proper subgroup of F (2) does not fix any number in $(0, 1)$ and (3) is not contained in any finite index subgroup of F . The conclusion was that every maximal subgroup of F containing H has infinite index in F and does not fix any number in $(0, 1)$. In this section we construct two explicit maximal subgroups of F containing H .

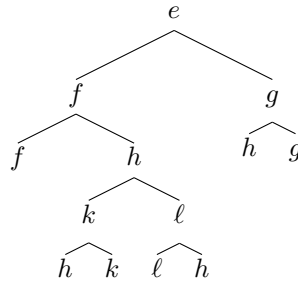
The idea in both constructions is similar. We start with the minimal tree $T_{\mathcal{L}(H)}^{\min}$ associated with $\mathcal{L}(H)$ and “extend” it to the minimal tree of some core-automaton \mathcal{L} which accepts a larger group. We apply Lemmas 10.9 and 10.10 to prove that \mathcal{L} is a core automaton. Then using the algorithm from Section 10.1, one can find a generating set of the closed group K accepted by \mathcal{L} . Since $\mathcal{L}(K)$ and \mathcal{L} coincide up to identification of vertices, if \mathcal{L} is chosen properly, then using Theorem 7.14, one can show that for any $f \notin K$, we have $\langle K, f \rangle = F$, which implies that K is a maximal subgroup of F .

The minimal tree $T_{\mathcal{L}(H)}^{\min}$ for $H = \langle x_0, x_1 x_2 x_1^{-1} \rangle$ is the following. It was given in Remark 4.11 with different labels.



Example 10.12. The group $K = \langle x_0, x_1 x_2 x_1^{-1}, x_1^2 x_2^{-1} \rangle$ is a maximal subgroup of F which contains H .

Proof. Given the group K , one could start the proof by finding the core $\mathcal{L}(K)$. We do not do so and instead explain how we got the group K . We let \mathcal{L} be the folded-automaton given by the following minimal tree $T_{\mathcal{L}}^{\min}$.



The tree $T_{\mathcal{L}}^{\min}$ can be viewed as an “extension” of $T_{\mathcal{L}(H)}^{\min}$. Indeed, $T_{\mathcal{L}(H)}^{\min}$ is a rooted subtree of $T_{\mathcal{L}}^{\min}$. An immediate implication is that H is accepted by the folded automaton \mathcal{L} . Indeed, every diagram accepted by $\mathcal{L}(H)$ is in particular accepted by \mathcal{L} .

Next, we prove that \mathcal{L} is a core-automaton. Indeed, \mathcal{L} satisfies condition (1) from Lemma 10.9 by definition. It is easy to verify that it satisfies condition (2). To check condition (3) we use Lemma 10.10. Let \mathcal{P} be the semigroup presentation given by \mathcal{L} and let S be the

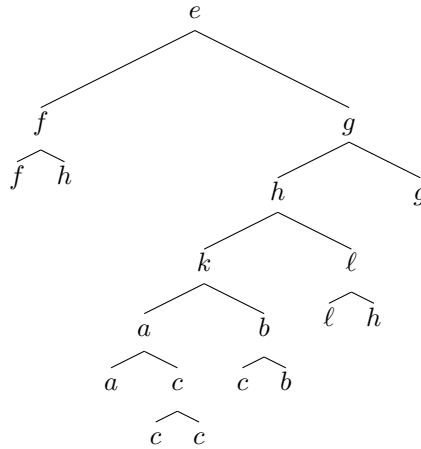
semigroup with presentation \mathcal{P} . Let u and v be paths on $T_{\mathcal{L}}^{\min}$ such that $\text{lab}(u^+) \equiv \text{lab}(v^+)$. If u^+ and v^+ are vertices of the rooted subtree $T_{\mathcal{L}(H)}^{\min}$, then condition (3) of Lemma 10.9 holds as $\mathcal{L}(H)$ is a core automaton and any diagram accepted by $\mathcal{L}(H)$ is accepted by \mathcal{L} . Thus, we can assume that $u \equiv 0100$ or $u \equiv 0101$. In the first case, $\text{lab}(u^+) \equiv h$. By transitivity arguments, it suffices to check that the condition in Lemma 10.10 holds for u and $v \equiv 01$ (we could choose v to be any path to a vertex with label h in $T_{\mathcal{L}(H)}^{\min}$). In the notations of Lemma 10.10, we have $(p_u, q_u) \equiv (f, k\ell g)$ and $(p_v, q_v) \equiv (f, g)$. Thus, $p_u = q_u$ in the semigroup S . Similarly, $q_v \equiv g = hg = (k\ell)g \equiv q_u$ in S . The case where $u \equiv 0101$ can be treated in a similar way. Thus, \mathcal{L} is a core automaton.

Now, one can apply the simplified algorithm from Section 10.1 to find a generating set of the closed group K accepted by \mathcal{L} . One gets that $K = \langle x_0, x_1 x_2 x_1^{-1}, x_1^2 x_2^{-1} \rangle$.

To prove that K is a maximal subgroup of F we use an argument similar to the one in Theorem 9.1. Namely, we prove that for all $f \in F \setminus K$, $\text{Cl}(\langle K, f \rangle) \supseteq [F, F]$. The proof is identical to the proof of Lemma 9.2, so we do not repeat it. We note that the group H , and thus K and any subgroup of F containing it, satisfies conditions (2) and (3) in Theorem 7.14. Indeed, H is not contained in any finite index subgroup of F and for $z = x_1 x_2 x_1^{-1} \in H$, z fixes the dyadic fraction $\alpha = .101$, $z'(\alpha^-) = 1$ and $z'(\alpha^+) = 2$. Thus, for any $f \notin K$, the group $\langle K, f \rangle$ satisfies conditions (1)-(3) of Theorem 7.14. It follows that for any $f \notin K$, $\langle K, f \rangle = F$, which implies that K is a maximal subgroup of F . \square

Example 10.13. The group $K = \langle x_0, x_1 x_2 x_1^{-1}, x_1^2 x_2 x_1^{-3}, x_1^3 x_2 x_1^{-4} \rangle$ is a maximal subgroup of F containing H .

Proof. We let \mathcal{L} be the folded-automaton given by the following minimal tree $T_{\mathcal{L}}^{\min}$.



$T_{\mathcal{L}(H)}^{\min}$ is a rooted subtree of $T_{\mathcal{L}}^{\min}$. Thus, H is accepted by \mathcal{L} .

The proof that \mathcal{L} is a core automaton can be done, as in Example 10.12, by considering all pairs of paths u and v on $T_{\mathcal{L}}^{\min}$ with $\text{lab}(u^+) \equiv \text{lab}(v^+)$. Alternatively, we note that to construct the tree $T_{\mathcal{L}}^{\min}$, we started with minimal trees $T_{\mathcal{L}(H)}^{\min}$ and $T_{\mathcal{L}(F)}^{\min}$ associated with the cores $\mathcal{L}(H)$ and $\mathcal{L}(F)$ respectively (see Remark 7.2), and identified the root of $T_{\mathcal{L}(F)}^{\min}$ with the leaf $(100)^+$ of $T_{\mathcal{L}(H)}^{\min}$. Thus, if u and v are paths on \mathcal{L} with $\text{lab}(u^+) \equiv \text{lab}(v^+)$, then

u^+ and v^+ both belong to the rooted subtree $T_{\mathcal{L}(H)}^{\min}$ or both belong to the subtree of $T_{\mathcal{L}}^{\min}$ rooted at $(100)^+$. In the first (resp. second) case, condition (3) of Lemma 10.9 holds for u and v since the condition holds in $T_{\mathcal{L}(H)}^{\min}$ (resp. $T_{\mathcal{L}(F)}^{\min}$).

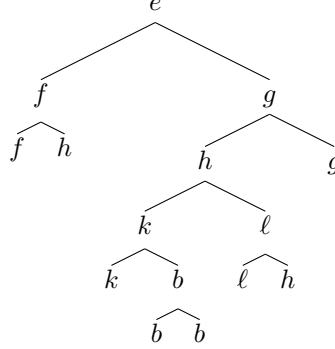
An application of the algorithm in Section 10.1 shows that the group K accepted by \mathcal{L} is the one given in the example. As in Example 10.12, any subgroup of F containing K satisfies conditions (2) and (3) in Theorem 7.14. Thus, the following lemma completes the proof of maximality of K in F .

Lemma 10.14. Let $f \in F \setminus K$. Then $\text{Cl}(\langle K, f \rangle) \supseteq [F, F]$.

Proof. We let $M = \langle K, f \rangle$. The proof is similar, but more complicated than the proof of Lemma 9.1. The core $\mathcal{L}(K)$ is described by the minimal tree $T_{\mathcal{L}}^{\min}$ given above. We note that every edge in $\mathcal{L}(K)$ is the top edge of some positive cell and that $\mathcal{L}(K)$ has a unique left boundary edge and a unique right boundary edge. Thus, as in the proof of Lemma 9.1, there is a surjective morphism ϕ from $\mathcal{L}(K)$ to $\mathcal{L}(M)$. In addition, for some pair of distinct inner edges $e_1 \neq e_2$ in $\mathcal{L}(K)$, we must have $\phi(e_1) = \phi(e_2)$ in $\mathcal{L}(M)$. The inner edges of $\mathcal{L}(K)$ are h, ℓ, k, a, b, c . To get the result, we have to go over all choices of inner edges $e_1 \neq e_2$ in $\mathcal{L}(K)$ and show that if $\phi(e_1) = \phi(e_2)$ in $\mathcal{L}(M)$, then $\phi(h) = \phi(\ell) = \phi(k) = \phi(a) = \phi(b) = \phi(c)$, so that there is only one inner edge in $\mathcal{L}(M)$. As there are 15 possible choices of pairs $e_1 \neq e_2$, we describe only some of them. The other cases can be verified in a similar way.

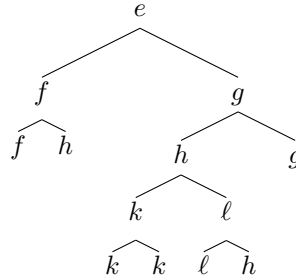
First, assume that $e_1 = h$ and $e_2 = k$, so that $\phi(h) = \phi(k)$. As in the proof of Lemma 9.1, we consider the implications to the image of other inner edges in $\mathcal{L}(H)$. In $T_{\mathcal{L}(H)}^{\min}$, a vertex labeled h has a left child labeled k which has a left child labeled a . Thus, the identification of h and k under ϕ implies that k and a are also identified. Considering caret in $T_{\mathcal{L}(K)}^{\min}$ with top vertices labeled h, k and a , we see that the right children must all be identified under ϕ . Thus, $\phi(\ell) = \phi(b) = \phi(c)$. Again, considering right children, we get that $\phi(h) = \phi(b) = \phi(c)$. All together, we get that $\phi(h) = \phi(\ell) = \phi(k) = \phi(a) = \phi(b) = \phi(c)$ as required. In a similar manner, one can show that if $e_1 = h$ and e_2 is any other inner edge of $\mathcal{L}(H)$, then $\mathcal{L}(M)$ has a unique inner edge. Indeed, similar arguments also work when $e_1 = \ell$ and $e_2 \neq \ell$. Thus, we only have to consider the case where $e_1, e_2 \in \{k, a, b, c\}$. We consider the case $e_1 = k$ and $e_2 = a$, the other cases being similar.

By assumption $\phi(k) = \phi(a)$. That implies that the right children of $\phi(a)$ and $\phi(k)$ satisfy $\phi(b) = \phi(c)$. We claim that there must be at least one more pair of edges of $\mathcal{L}(K)$ (other than $\{k, a\}$ and $\{b, c\}$) with the same image under ϕ . Otherwise, the core $\mathcal{L}(M)$ is given by the following minimal tree.



To get a contradiction it suffices to note that the above tree is not a minimal tree associated with a core-automaton. Indeed, consider the paths $u = 1001$ and $v = 10010$ on $T_{\mathcal{L}(M)}^{\min}$. Then, $\text{lab}(u^+) \equiv \text{lab}(v^+) \equiv b$. The pairs of words associated with the paths u and v in $T_{\mathcal{L}(M)}^{\min}$ are $(p_u, q_u) \equiv (fk, \ell g)$ and $(p_v, q_v) \equiv (fk, b\ell g)$. We claim that $q_u \neq q_v$ in the semigroup S with presentation \mathcal{P} associated with $\mathcal{L}(M)$. Indeed, if $\ell g = b\ell g$ in S , then, since ℓg is a 1-path in $\mathcal{L}(M)$, there is an $(\ell g, b\ell g)$ -diagram Δ over $\mathcal{L}(M)$. The diagram Δ implies that $\iota(\ell g) = \iota(b\ell g)$ in $\mathcal{L}(M)$. In particular, the vertices ℓ_- and b_- coincide in $\mathcal{L}(M)$, in contradiction to ℓ and b not being in the same connected component of $\Gamma(\mathcal{L}(M))$ (see Proposition 6.6).

Hence, at least one more pair of inner edges is identified in the transition from $\mathcal{L}(K)$ to $\mathcal{L}(M)$. If h or ℓ is one of the edges in the pair, we are done. Thus, we can assume that $\psi(k) = \psi(b)$. We claim that in this case, again, there must be another pair of identified edges. Otherwise, $\mathcal{L}(M)$ is described by the following associated minimal tree.



As above, one can show that $T_{\mathcal{L}(M)}^{\min}$ is not associated with a core-automaton to get the required contradiction by considering the pair of paths $u = 100$ and $v = 1000$ on $T_{\mathcal{L}(M)}^{\min}$. Therefore, at least one more pair of inner edges of $\mathcal{L}(K)$ is identified in $\mathcal{L}(M)$. In particular, either h or ℓ is identified with some other edge of $\mathcal{L}(K)$. As noted above, that implies that $\mathcal{L}(M)$ has a unique inner edge and completes the proof of the lemma. \square

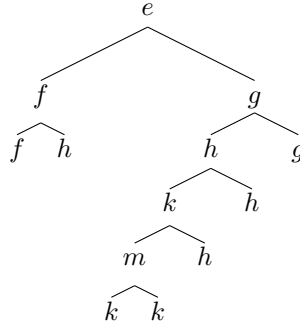
\square

10.3.B A maximal subgroup of F which acts transitively on the set \mathcal{D}

The following can be viewed as a strong counter example to Savchuk's problem [27, Problem 1.5].

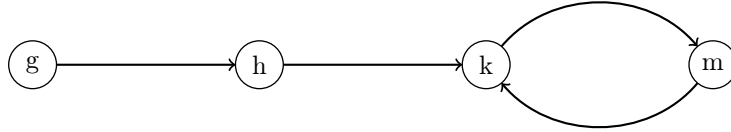
Lemma 10.15. The group $M = \langle x_0, x_1x_2x_1^{-3}, x_1x_2x_3x_2^{-3}x_1^{-1} \rangle$ is a maximal subgroup of infinite index in F which acts transitively on the set of finite dyadic fractions \mathcal{D} .

Proof. The core $\mathcal{L}(M)$ is given by the following associated minimal tree.



The group M was chosen originally to be the subgroup of F accepted by this core (the algorithm from Section 10.1 was used to find the given generating set). Hence $M = \text{Cl}(M)$. The proof that M is a maximal subgroup of F of infinite index is almost identical to the proof of maximality of K from Example 10.12. Indeed, the proof that $M[F, F] = F$ and that for each $f \notin M$, we have $[F, F] \leq \langle M, f \rangle$ is identical to the proof in Example 10.12. To prove that for any $f \notin M$ there is an element $z \in \langle M, f \rangle$ which fixes a finite dyadic fraction $\alpha \in (0, 1)$ and such that $z'(\alpha^+) = 2$ and $z'(\alpha^-) = 1$, we observe that the element $y = x_1x_2x_1^{-3}(x_1x_2x_3x_2^{-3}x_1^{-1})^{-1} \in M$ fixes the finite dyadic fraction $\beta = .111$ and has slope $y'(\beta^-) = 2$ and $y'(\beta^+) = 1$. Since for each $f \notin F$, $[F, F] \leq \langle M, f \rangle$, one can apply Lemma 8.2 to get the existence of an element $z \in \langle M, f \rangle$ as described.

It remains to note that every edge in $\mathcal{L}(M)$ is the top edge of some positive cell and that the graph $\Gamma(M)$ given below is connected and apply Theorem 6.5.



□

11 Solvable subgroups of Thompson group F

11.1 On the closure of solvable subgroups

In this section we consider the closure of solvable subgroups of F and prove the following theorem. The theorem follows from results of [4, 5, 7]. We prove each part separately below.

Theorem 11.1. Let H be a solvable subgroup of F of derived length n . Then the following assertions hold.

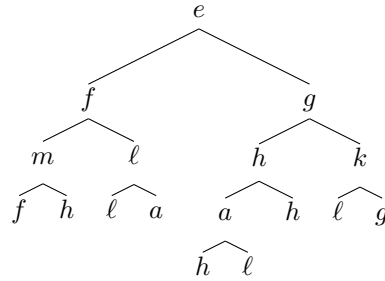
- (1) The action of H on the set of finite dyadic fractions \mathcal{D} has infinitely many orbits.
- (2) $\text{Cl}(H)$ is solvable of derived length n .

(3) If H is finitely generated then $\text{Cl}(H)$ is finitely generated.

Part (1) of Theorem 11.1 should be viewed in contrast with Theorem 9.1, where an elementary amenable subgroup $B \leq F$ which acts transitively on the set of finite dyadic fractions \mathcal{D} was constructed. Part (2) of Theorem 11.1 should be viewed in contrast with the following example.

Example 11.2. Let B_1 be the subgroup of F generated by $x = x_0x_1x_2x_3x_5^2(x_0x_1x_2x_4^3)^{-1}$ and $y = x_0^3x_2x_6(x_0x_1^2x_3x_5^2x_7)^{-1}$. Then B_1 is a copy of the Brin-Navas group. In particular, it is elementary amenable. The closure $\text{Cl}(B_1)$ contains a copy of Thompson group F . In particular, it is not elementary amenable [11].

Proof. The group B_1 is a copy of the Brin-Navas group, as realized in [6, Section 1.2.3]. To prove that the closure of B_1 contains a copy of F we consider the core $\mathcal{L}(B_1)$. The following is a minimal tree associated with $\mathcal{L}(B_1)$.



Let \mathcal{P} be the semigroup presentation associated with $\mathcal{L}(B_1)$ and let $p = p_{\mathcal{L}(B_1)} = e$ be the distinguished edge of $\mathcal{L}(B_1)$. Let S be the semigroup with presentation \mathcal{P} . We claim that there is a word w over the alphabet E of \mathcal{P} , such that (1) w divides p in S (i.e., there exist words a and b over E such that awb is equal to p in S) and (2) $w = ww$ in S . By [18, Theorem 25], that would imply that the diagram group $\text{DG}(\mathcal{L}(B_1), p) \cong \text{Cl}(B_1)$, contains a copy of Thompson group F . Let w be the edge a of $\mathcal{L}(B_1)$. Since w is a 1-path in $\mathcal{L}(B_1)$, it follows from Proposition 10.2(1) that w divides p in S . (Indeed, one can consider a word wq where q is a 1-path in $\mathcal{L}(B_1)$ with $\iota(q) = \tau(w)$ and $\tau(q) = \tau(\mathcal{L}(B_1))$.) It remains to note that $a = h\ell = h(\ell a) = (h\ell)a = aa$ in S . \square

Remark 11.3. Using similar arguments to those in the proof of Theorem 9.1, one can show that the closure of B_1 is a maximal subgroup of $B_1[F, F]$. Thompson group F is an infinite cyclic extension of $B_1[F, F]$.

In [4], solvable subgroups of $\text{PL}_o(I)$ are characterized by the towers associated with them. The following definition uses different notation than the one in [4].

Definition 11.4. Let G be a subgroup of $\text{PL}_o(I)$. A *tower* in G is a set of distinct intervals $T = \{(a_i, b_i) \mid i \in \mathcal{I}\}$ such that for each i , (a_i, b_i) is an orbital of some function in H and such that T is totally ordered with respect to inclusion.

The cardinality of a tower is said to be the *height of the tower*. The supremum of heights of all towers in G is called the *depth of G* . If G is the trivial subgroup, we say that G has depth 0.

Theorem 11.5. [4, Theorem 1.1] Let G be a subgroup of $\text{PL}_o(I)$. Then G is solvable of derived length n if and only if the depth of G is n .

Definition 11.6 ([4]). Let $G \leq \text{PL}_o(I)$. The group G admits a *transition chain* if there are elements g_1, g_2 in G with orbitals (a, b) and (c, d) respectively, such that $a < c < b < d$.

The following lemma follows immediately from Theorem 1.1, Lemma 1.4 and Remark 4.1 of [4].

Lemma 11.7 ([4]). Let G be a solvable subgroup of $\text{PL}_o(I)$. Then G does not admit transition chains. Moreover, if (a, b) and (c, d) are distinct orbitals of elements $g, h \in G$ such that $(a, b) \cap (c, d) \neq \emptyset$, then either $a < c < d < b$ or $c < a < b < d$.

We apply Theorem 11.5 and Lemma 11.7 to prove the following lemma. This is part (1) of Theorem 11.1.

Lemma 11.8. Let H be a solvable subgroup of F . Then the action of H on the set of finite dyadic fractions \mathcal{D} has infinitely many orbits.

Proof. Let n be the derived length of H . By Theorem 11.5, H has a maximal tower $T = \{(a_i, b_i) : i = 1, \dots, n\}$ of height n . We denote by (a, b) the smallest orbital in the tower. Let $h \in H$ be a function with orbital (a, b) . Let f in H be a function which maps a number $x \in (a, b)$ to a number $y \neq x$ in this interval. We claim that (a, b) is an orbital of f and in particular that f fixes a and b . Indeed, let (c, d) be the orbital of f containing x and assume by contradiction that (c, d) does not coincide with (a, b) . Replacing f by f^{-1} if necessary, we can assume that (c, d) is a push-up orbital of f . By Lemma 11.7, either (c, d) is strictly contained in (a, b) , or we have $c < a < b < d$. Since T is a maximal tower of H , we must have $c < a < b < d$. In particular, $a \in (c, d)$. Since $a < x$ and (c, d) is a push-up orbital of f we have $a < f(a) < f(x) = y < b$. We consider the element h^f . It has an orbital $(f(a), f(b))$. Since $f(a) \in (a, b)$, by Lemma 11.7, the orbital $(f(a), f(b))$ must be strictly contained in (a, b) . That gives a contradiction to the tower T being a tower of H of maximal height. Thus, (a, b) is an orbital of f .

Let K be the subgroup of H of all elements which fix the numbers a and b . One can naturally map K onto a subgroup K' of $\text{PL}_o(I)$ where all functions have support in $[a, b]$, by sending a function $k \in K$ to the function $k' \in K'$ which coincides with k on $[a, b]$ and is the identity elsewhere. Let $\alpha_1, \alpha_2 \in (a, b)$ be two finite dyadic fractions. The above argument shows that α_1 and α_2 belong to the same orbit of the action of H on \mathcal{D} if and only if they belong to the same orbit of the action of K' on \mathcal{D} . The maximality of the tower T implies that any non-trivial element in K' has orbital (a, b) .

We claim that K' is cyclic. Indeed, let $k_1 \in K'$ be an element with slope 2^m at a^+ for minimal $m > 0$. If $k_2 \in K'$, has slope 2^{m_2} at a^+ then m divides m_2 . Hence, $k' = k_1^{\frac{m_2}{m}} k_2^{-1}$ has slope 1 at a^+ . It follows that k' fixes a right neighborhood of a and as such (a, b) is not an orbital of k' . Thus, k' is the trivial element of K' which implies that $K' = \langle k_1 \rangle$. It is obvious that the action of $\langle k_1 \rangle$ on $(a, b) \cap \mathcal{D}$ has infinitely many orbits. Thus, the action of H on \mathcal{D} has infinitely many orbits. \square

Part (2) of Theorem 11.1 follows immediately from results of Bleak [5]. For a subgroup G of $\text{PL}_o(I)$, Bleak defined the split group $S(G)$ of G . We define $S(G)$ using different

terminology to emphasize the relation between $S(G)$ and the closure $\text{Cl}(G)$ in case G is a subgroup of F .

Definition 11.9. Let f be a function in $\text{PL}_o(I)$. If f fixes a number $\alpha \in (0, 1)$, then the following functions $f_1, f_2 \in F$ are called *general components* of the function f , or *general components of f at α* .

$$f_1(t) = \begin{cases} f(t) & \text{if } t \in [0, \alpha] \\ t & \text{if } t \in [\alpha, 1] \end{cases} \quad f_2(t) = \begin{cases} t & \text{if } t \in [0, \alpha] \\ f(t) & \text{if } t \in [\alpha, 1] \end{cases}$$

Note that the only difference between Definition 5.1 and Definition 11.9 is that general components are taken with respect to fixed points which are not necessarily finite dyadic.

Definition 11.10 (Bleak [5]). Let $G \leq \text{PL}_o(I)$. The *split group* of G is the subgroup of $\text{PL}_o(I)$ generated by all elements of G and all general components of elements in G .

Remark 11.11. A simple adaptation of the proof of Lemma 5.5 shows that $S(G)$ can be defined as the subgroup of $\text{PL}_o(I)$ of all piecewise- G functions. It follows that $S(S(G)) = S(G)$. We also note that the orbits of the action of G on the interval $[0, 1]$ coincide with the orbits of the action of $S(G)$.

Theorem 11.12. [5, Lemma 4.5, Corollary 4.6] Let G be a solvable subgroup of $\text{PL}_o(I)$. Then (a, b) is an orbital of an element in G if and only if it is an orbital of an element in $S(G)$. In particular, by Theorem 11.5, the derived length of G is equal to the derived length of the split group $S(G)$.

For a subgroup $H \leq F$, by Theorem 5.6, we have $H \leq \text{Cl}(H) \leq S(H)$. Thus, Theorem 11.12 implies the following.

Corollary 11.13 (Theorem 11.1(2)). Let $H \leq F$ be a solvable subgroup of F . Then the closure $\text{Cl}(H)$ is solvable of the same derived length.

To prove Theorem 11.1(3) we need the following lemma.

Lemma 11.14. [7, Lemma 3.1] Let $G \leq \text{PL}_o(I)$ be a finitely generated subgroup. Then the set of breakpoints of elements in G intersects finitely many orbits of the action of G on $[0, 1]$.

Proposition 11.15. Let $G \leq \text{PL}_o(I)$ be a finitely generated solvable subgroup. Then the split group $S(G)$ is finitely generated.

Proof. Assume that G is solvable of derived length n . By Theorem 11.12, orbitals of elements in G coincide with orbitals of elements in $S(G)$. We say that an orbital (a, b) of some function in G (equiv. $S(G)$) is *minimal* in G if there is no function in G with an orbital strictly contained in (a, b) . Note that if (a, b) is a minimal orbital in G and $g \in G$ then $(g(a), g(b))$ is also a minimal orbital in G . Note also that by Theorem 11.5 every orbital of an element in G contains some minimal orbital. In addition, by Theorem 11.5 and Lemma 11.7, a minimal orbital in G is contained in at most n orbitals of elements of G .

The proof of [5, Lemma 4.7] shows that the split group $S(G)$ is generated by a set of functions $B = \{g_i \mid i \in \mathcal{I}\}$ such that for each i , g_i has a single orbital (a_i, b_i) and such that the following conditions hold.

- (1) For every $i \neq j$ and every $g \in G$ we have $(g(a_i), g(b_i)) \neq (a_j, b_j)$. In particular, for every $i \neq j$, $(a_i, b_i) \neq (a_j, b_j)$.
- (2) If (a_i, b_i) is not a minimal orbital in G then it contains an orbital (a_j, b_j) for some j , such that (a_j, b_j) is minimal in G .

We claim that the generating set B must be finite. Otherwise, we let $\mathcal{J} \subseteq I$ be the subset of all indexes j such that the orbitals (a_j, b_j) are minimal orbitals in G . Let C be the set of orbitals (a_j, b_j) for $j \in \mathcal{J}$. We claim that C must be infinite. Indeed, by condition (2) for the set B one can map each generator g_i in B to some orbital (a_j, b_j) in C such that the orbital (a_j, b_j) is contained in (a_i, b_i) . By condition (1), the preimage of each orbital in C is a subset of size at most n of B . Thus, C is infinite.

By Theorem 11.12, each orbital (a_j, b_j) in C is an orbital of some element h_j in G . We note that h_j must have a breakpoint $x_j \in (a_j, b_j)$. Indeed, a function in $\text{PL}_o(I)$ cannot be linear on any of its orbitals. We let E be the set of breakpoints x_j , $j \in \mathcal{J}$. We note that if $j \neq k$ in \mathcal{J} , then $x_j \neq x_k$. Indeed, (a_j, b_j) and (a_k, b_k) are distinct minimal orbitals in G . Hence by Lemma 11.7, they are disjoint. Moreover, x_j and x_k do not belong to the same orbit of the action of G on $[0, 1]$. Indeed, if $g \in G$, then $g(x_j) \in (g(a_j), g(b_j)) \neq (a_k, b_k)$ by condition (1). Since $(g(a_j), g(b_j))$ and (a_k, b_k) are minimal orbitals in G , they are disjoint, and so $g(x_j) \neq x_k$. Thus, the set E is a set of breakpoints of elements of G which intersects infinitely many orbits of the action of G on $[0, 1]$. We get a contradiction to the finite generation of G by Lemma 11.14. \square

The proof of Theorem 11.1(3) requires a slight adaptation of the proof of Proposition 11.15.

Definition 11.16. Let $f \in F$ and assume that $f \in F$ fixes finite dyadic fractions $a, b \in [0, 1]$ such that $a < b$. We say that (a, b) is a *dyadic-orbital* of f if f does not fix any finite dyadic fraction in (a, b) .

By [16, Corollary 2.5] (See also [27]), if f fixes an irrational number x in $(0, 1)$ then f fixes an open neighborhood of x . Hence, if (a, b) is a dyadic-orbital of f , which is not an orbital of f , then there are finitely many rational non-dyadic numbers $x_1 < \dots < x_n$ in (a, b) such that $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, b)$ are orbitals of f . Thus, a dyadic-orbital of $f \in F$ can be defined as a minimal sequence of consecutive “adjacent” orbitals $(a_1, a_2), (a_2, a_3) \dots, (a_k, a_{k+1})$ such that a_1, a_{k+1} are finite dyadic.

Proof of Theorem 11.1(3). The proof of the theorem is almost identical to the proof of Proposition 11.15. Indeed, one only has to replace every occurrence of the word “orbital” in the proof of Proposition 11.15 with the term “dyadic-orbital” and replace the group $S(G)$ with the group $\text{Cl}(G)$. A detailed proof requires an adaptation of Lemma 11.7 and the first statement of Theorem 11.12 to the case where G is a subgroup of F and dyadic-orbitals are considered instead of orbitals. It also requires a proof that the conclusion of [5, Lemma 3.12] (which is used in the proof of [5, Lemma 4.7]) holds for a solvable subgroup $G \leq F$ with “orbital” replaced by “dyadic-orbital”. The adaptations are not hard but as we do not wish to repeat here paper [5], we omit them. \square

11.2 Characterization of solvable subgroups $H \leq F$ in terms of the core $\mathcal{L}(H)$

Recall that if (a, b) is a push-down orbital of a function $g \in \text{PL}_o(I)$ then the slope $g'(a^+) < 1$. By [16, Corollary 2.5], if (a, b) is an orbital of a function $f \in F$, then a and b must be rational numbers. The proof of the following lemma is similar to the proof of Lemma 2.6 and is left as an exercise to the reader.

Lemma 11.17. Let $f \in F$ be a function with a push-down orbital (a, b) . Then the reduced diagram Δ of f has a pair of branches $u \rightarrow us$ for some finite binary words u and s such that s contains the digit 0 and $a = .us^{\mathbb{N}}$.

Recall (Section 4) that a path on the core $\mathcal{L}(H)$ always starts from the distinguished edge $p_{\mathcal{L}(H)} = q_{\mathcal{L}(H)}$ whereas a trail on $\mathcal{L}(H)$ can start from any edge. We rarely distinguish between a trail and its label, but when describing a trail we are careful to mention its initial edge.

Definition 11.18. Let $H \leq F$. An edge e of the core $\mathcal{L}(H)$ is a *periodic edge* if there is a trail s with initial edge e and terminal edge e such that the label s contains the digit 0.

Definition 11.19. Let $H \leq F$ and let e be a periodic edge of $\mathcal{L}(H)$. We let \mathcal{P}_e be the set of all non-trivial trails from e to itself which do not visit any edge of the core more than twice. By assumption, \mathcal{P}_e is not empty. We denote by s_e the smallest trail in \mathcal{P}_e with respect to the lexicographic order on the set of non-empty finite binary words $\{0, 1\}^+$ (where 0 is taken to be smaller than 1). We call s_e the *optimal trail from e to itself* and note that s_e must contain the digit 0 by the definition of periodic edges.

Proposition 11.20. Let e_1 and e_2 be periodic edges of $\mathcal{L}(H)$. Then the following conditions are equivalent.

- (1) There is a trail q from e_1 to e_2 on $\mathcal{L}(H)$ and a trail s from e_1 to itself such that $.q > .s^{\mathbb{N}}$.
- (2) There is a trail q from e_1 to e_2 such that

$$q \equiv u1w_1 \quad \text{and} \quad s_{e_1} \equiv u0w_2,$$

where u, w_1, w_2 are finite binary words and the sub-trail of q given by the suffix w_1 is a simple trail; i.e., it does not visit any edge in $\mathcal{L}(H)$ more than once.

- (3) There is a trail q from e_1 to e_2 such that $.q > .s_{e_1}^{\mathbb{N}}$.

Proof. It is clear that (2) implies (3). Indeed, the trail q from (2) satisfies $.q > .s_{e_1}^{\mathbb{N}}$. It is also obvious that (3) implies (1). To prove that (1) implies (2), let q and s be trails as described in condition (1). We can assume that s is not a prefix of q . Otherwise, $q \equiv sq'$. Since s labels a trail from e_1 to itself, q' is a trail from e_1 to e_2 . In addition, since $.q = .sq' > .s^{\mathbb{N}}$, we have $.q' > .s^{\mathbb{N}}$. Thus, q can be replaced by q' .

Now, consider the binary words q and $s^{\mathbb{N}}$ as paths on the complete infinite binary tree. We let u be the longest common prefix of q and $s^{\mathbb{N}}$. The assumption $.q > .s^{\mathbb{N}}$ implies that $q \equiv u1w_1$ and $s^{\mathbb{N}} \equiv u0w_2$ where w_1 is a possibly empty suffix of q and w_2 is an infinite suffix of $s^{\mathbb{N}}$. We can assume that the sub-trail of q given by the suffix w_1 is a simple trail.

Otherwise, one can replace w_1 by a simple trail with the same initial and terminal vertices and condition (1) would still hold for q and s .

Since s is not a prefix of q , and as such, not a prefix of u ; u must be a strict prefix of s . Thus, $s \equiv u0w_2$ for some possibly empty suffix w_2 . If the trail s from e_1 to itself does not visit any edge of $\mathcal{L}(H)$ more than twice, then by the choice of the optimal trail s_{e_1} , s_{e_1} is smaller or equal to s in the lexicographic order on $\{0,1\}^+$. In that case, there are two options.

(1) s_{e_1} is not a prefix of s . In that case, we have $s_{e_1} \equiv a0b_1$ and $s \equiv a1b_2$ where a is the longest common prefix of s_{e_1} and s and b_1 and b_2 are finite binary words. That implies that $.q > .s^{\mathbb{N}} > .s_{e_1}^{\mathbb{N}}$. Thus, the trails q and s_{e_1} satisfy condition (1). As above, that implies that we can assume (by replacing them by other trails if necessary), that if v is the longest common prefix of q and s_{e_1} then $q \equiv v1w'_1$ and $s_{e_1} \equiv v0w'_2$ where the sub-trail of q given by the suffix w'_1 is a simple trail on $\mathcal{L}(H)$. Then condition (2) is satisfied.

(2) $s \equiv s_{e_1}b$ for some finite binary word b . We claim that in that case $b \equiv \emptyset$, so that $s \equiv s_{e_1}$ and we are done by the forms of q and s found above. Indeed, since s_{e_1} visits the edge e_1 exactly twice and $s \equiv s_{e_1}b$ terminates on the edge e , b must be empty, or else, s would visit e_1 at least 3 times.

Thus, we can assume that the trail s visits some edge e at least 3 times. Since $s \equiv u0w_2$, there are two options.

(1) The sub-trail u visits an edge e at least twice. Thus, by cutting out a cycle from the trail u , one can replace it by a shorter trail u' which visits every edge visited by u at most as many times and visits the edge e less times than the trail u . We note that replacing u by u' in both trails $q \equiv u1w_1$ and $s \equiv u0w_2$ would not affect condition (1). Thus, we are done by induction on the number of times the trail s visits an edge which was already visited twice.

(2) The trail w_2 visits an edge e at least twice. In which case, we can replace w_2 by a shorter trail by cutting out a cycle from w_2 . Again, condition (1) is not affected and we are done by induction. \square

Definition 11.21. Let H be a subgroup of F . We define a directed graph $\mathcal{P}(H)$ as follows. The vertex set of $\mathcal{P}(H)$ is the set of periodic edges of $\mathcal{L}(H)$. If e_1 and e_2 are periodic edges, then there is a directed edge from e_1 to e_2 in $\mathcal{P}(H)$ if and only if the equivalent conditions in Proposition 11.20 are satisfied for e_1 and e_2 .

The *length* of a directed path in $\mathcal{P}(H)$ is the number of vertices in the path.

Lemma 11.22. Let H be a subgroup of F such that $\text{Cl}(H)$ does not admit transition chains. If there is a directed path of length n in $\mathcal{P}(H)$ then $\text{Cl}(H)$ admits a tower $T = \{(a_i, b_i), i = 1, \dots, n\}$ of height n .

Proof. Let $e_1 \rightarrow e_2 \rightarrow \dots \rightarrow e_n$ be a directed path in $\mathcal{P}(H)$. Then by Condition (3) in Proposition 11.20, for each $i = 1, \dots, n-1$ there is a trail q_i on $\mathcal{L}(H)$ from e_i to e_{i+1} such that $.q_i > .s_{e_i}^{\mathbb{N}}$. Let u be a path on $\mathcal{L}(H)$ with terminal edge e_1 . For each $i \in \{1, \dots, n\}$, we let $v_i \equiv uq_1 \dots q_{i-1}$ and note that v_i is a path on $\mathcal{L}(H)$ with terminal edge e_i .

Since the edges e_i are all periodic edges, for each i , v_i and $v_i s_{e_i}$ are paths on $\mathcal{L}(H)$ with terminal edge e_i . By Lemma 6.1, there is a function $f_i \in \text{Cl}(H)$ with a pair of branches $v_i \rightarrow v_i s_{e_i}$. In particular, f_i fixes the rational fraction $.v_i s_{e_i}^{\mathbb{N}}$ but does not fix a right neighborhood

of it. Indeed, s_{e_i} contains the digit 0, so $.v_i s_{e_i}^{\mathbb{N}}$ is not the right endpoint of $[v_i]$. Since f_i is linear on $[v_i]$, it does not fix any number in $[v_i]$ other than $.v_i s_{e_i}^{\mathbb{N}}$, and in particular, it does not fix any number in $(.v_i s_{e_i}^{\mathbb{N}}, .v_i 1^{\mathbb{N}}]$. Thus, f_i has orbital (a_i, b_i) where $a_i = .v_i s_{e_i}^{\mathbb{N}}$ and $b_i > .v_i 1^{\mathbb{N}}$.

Since for each $i = 1, \dots, n-1$, $v_{i+1} \equiv v_i q_i$ and $.q_i > .s_{e_i}^{\mathbb{N}}$, we have

$$a_{i+1} = .v_{i+1} s_{e_{i+1}}^{\mathbb{N}} \geq .v_{i+1} = .v_i q_i > .v_i s_{e_i}^{\mathbb{N}} = a_i.$$

Thus, $a_1 < a_2 < \dots < a_n$.

Since for each $i = 1, \dots, n$, $b_i > .v_i 1^{\mathbb{N}}$, for each $i = 1, \dots, n-1$, we have $a_{i+1} = .v_{i+1} s_{e_{i+1}}^{\mathbb{N}} = .v_i q_i s_{e_{i+1}}^{\mathbb{N}} < .v_i 1^{\mathbb{N}} < b_i$. It follows that for each $i = 1, \dots, n-1$, $a_{i+1} \in (a_i, b_i)$. Thus, since $\text{Cl}(H)$ does not admit transition chains, we must have $b_1 \geq b_2 \geq \dots \geq b_n$. Hence, $(a_1, b_1) \supset (a_2, b_2) \supset \dots \supset (a_n, b_n)$ and so $\{(a_i, b_i) \mid i = 1, \dots, n\}$ is a tower of height n in $\text{Cl}(H)$. \square

Let $G \leq \text{PL}_o(I)$. A tower $T = \{(a_i, b_i) \mid i \in \mathcal{I}\}$ in G is *good* if for distinct i and j , either $a_i < a_j < b_j < b_i$ or $a_j < a_i < b_i < b_j$. A good tower is a weaker version of an *exemplary tower* defined in [4].

Lemma 11.23. Let H be a subgroup of F which admits a good tower of height n . Then there is a directed path of length n in the graph $\mathcal{P}(H)$.

Proof. Let $T = \{(a_i, b_i) \mid i = 1, \dots, n\}$ be a good tower in H and assume that the chain of intervals (a_i, b_i) , $i = 1, \dots, n$ is decreasing. In particular, $a_1 < a_2 < \dots < a_n$ and $b_n < b_{n-1} < \dots < b_1$. For each i , let $h_i \in H$ be an element with orbital (a_i, b_i) . We assume that for all i , (a_i, b_i) is a push-down orbital of h_i , otherwise, one can replace h_i by h_i^{-1} . We use the elements h_i to construct a directed path of length n in $\mathcal{P}(H)$.

For $i = 1$, by Lemma 11.17, the reduced diagram Δ_1 of h_1 has a pair of branches of the form $u_1 \rightarrow u_1 s_1$, where s_1 contains the digit 0 and $a_1 = .u_1 s_1^{\mathbb{N}}$. Since Δ_1 is reduced, the words u_1 and $u_1 s_1$ label paths on $\mathcal{L}(H)$ with the same terminal edge. Let $e_1 = u_1^+ = (u_1 s_1)^+$ and note that e_1 is a periodic edge of $\mathcal{L}(H)$.

Since h_1 is linear on $[u_1]$ it does not fix any number in $[u_1]$ apart from a_1 . Therefore, the interval $(a_1, .u_1 1^{\mathbb{N}}]$ (which is not empty as s_1 contains the digit 0), is contained in the orbital (a_1, b_1) . Let $y_1 \in (a_1, .u_1 1^{\mathbb{N}}]$. Since $b_2, y_1 \in (a_1, b_1)$ and (a_1, b_1) is an orbital of h_1 , for some $k_1 \in \mathbb{Z}$ we have $h_1^{k_1}(b_2) < y_1$. In particular $h_2^{h_1^{k_1}}$ has orbital $(h_1^{k_1}(a_2), h_1^{k_1}(b_2))$ contained in (a_1, y_1) .

We replace all functions h_i , $i > 1$ in the sequence (h_i) , $i = 1, \dots, n$, by the conjugates $h_i^{h_1^{k_1}}$ and each orbital (a_i, b_i) , $i > 1$ with the orbital $(h_1^{k_1}(a_i), h_1^{k_1}(b_i))$. We denote the resulting sequence of functions again by (h_i) , $i = 1, \dots, n$. Similarly, we will refer to the orbitals in the new sequence of orbitals again by (a_i, b_i) , $i = 1, \dots, n$. Note that this sequence is decreasing and forms a good tower in H .

Now, for $i = 2$, by Lemma 11.17, the reduced diagram Δ_2 of h_2 has a pair of branches of the form $u_2 \rightarrow u_2 s_2$, where s_2 contains the digit 0 and $a_2 = .u_2 s_2^{\mathbb{N}}$. Since Δ_2 is reduced, the words u_2 and $u_2 s_2$ label paths on $\mathcal{L}(H)$ with the same terminal edge. Let $e_2 = u_2^+ = (u_2 s_2)^+$ and note that e_2 is a periodic edge of $\mathcal{L}(H)$. Since f_2 is linear on $[u_2]$ and fixes a_2 , the (non-empty) interval $(a_2, .u_2 1^{\mathbb{N}}]$ is contained in $(a_2, b_2) \subseteq (a_1, y_1)$ and thus, it is contained in the

interior of $[u_1]$. That implies that $[u_2] \subset [u_1]$. (Indeed, the right endpoint of the dyadic interval $[u_2]$ is contained in the interior of $[u_1]$.) Since $a_1 < a_2$, for a large enough $m \in \mathbb{N}$, every number in the interval $[u_2 s_2^m]$ is greater than a_1 . Since $[u_2 s_2^m] \subseteq [u_2] \subseteq [u_1]$ we have $u_2 s_2^m \equiv u_1 q_1$ for some suffix q_1 . Since u_1 labels a path on $\mathcal{L}(H)$ with terminal edge e_1 , and the path $u_2 s_2^m$ on $\mathcal{L}(H)$ terminates on e_2 , the word q_1 labels a trail from e_1 to e_2 . In addition, $.u_2 s_2^m = .u_1 q_1 > a_1 = .u_1 s_1^{\mathbb{N}}$. Thus, the trail q_1 from e_1 to e_2 and the trail s_1 from e_1 to itself satisfy condition (1) from Proposition 11.20. It follows that there is a directed edge in $\mathcal{P}(H)$ from e_1 to e_2 .

Continuing in this manner, we get periodic edges e_1, e_2, \dots, e_n in $\mathcal{L}(H)$ such that for each $i = 1, \dots, n-1$, there is a directed edge in $\mathcal{P}(H)$ from e_i to e_{i+1} . That completes the proof of the lemma. \square

Theorem 11.24. Let $H \leq F$. Then H is solvable if and only if the graph $\mathcal{P}(H)$ does not contain arbitrarily long positive paths. If H is solvable then the derived length of H is the maximal length of a positive path in $\mathcal{P}(H)$.

Proof. Assume that H is solvable. By Corollary 11.13, $\text{Cl}(H)$ is also solvable. In particular, by Lemma 11.7, $\text{Cl}(H)$ does not admit transition chains. Therefore, if $\mathcal{P}(H)$ contains arbitrarily long directed paths, then by Lemma 11.22, $\text{Cl}(H)$ admits arbitrarily high towers, in contradiction to Theorem 11.5. In the other direction, assume that H is not solvable. If H admits a transition chain then by [4, Lemma 2.11], H admits an infinite good tower. Otherwise, by [4, Lemma 2.12], every tower in H is good. It follows from Theorem 11.5 that in that case, H admits arbitrarily high good towers. In both cases, by Lemma 11.23, there are arbitrarily long directed paths in $\mathcal{P}(H)$.

The claim about the derived length of H follows in a similar way from Theorem 11.5, Corollary 11.13 and Lemmas 11.7, 11.22 and 11.23. \square

A finite directed graph contains arbitrarily long directed paths if and only if the graph contains a positive cycle. Thus, we have the following.

Corollary 11.25. Let H be a subgroup of F with finite core $\mathcal{L}(H)$. Then H is solvable if and only if there is no directed cycle in $\mathcal{P}(H)$.

We remark that if $H \leq F$ is finitely generated, then Corollary 11.25 gives a simple algorithm for deciding if H is solvable or not. Indeed, one has to construct the directed graph $\mathcal{P}(H)$, and check whether or not it contains a cycle.

Note that if H is finitely generated, there are finitely many edges in the core $\mathcal{L}(H)$. For each edge e in $\mathcal{L}(H)$, there are finitely many trails with initial edge e which do not visit any edge more than twice. Thus, one can check if e is a periodic edge and if it is, find the optimal trail s_e . If e is periodic then to find all outgoing edges of e in $\mathcal{P}(H)$, one has to consider all trails of the form $q \equiv u_1 w$ with initial edge e where u_0 is a prefix of s_e and the sub-trail of q given by the suffix w is simple. Clearly, there are finitely many trails to consider.

In fact, it is not too hard to show that the construction of the graph $\mathcal{P}(H)$ can be done in $O(n^3)$ time where n is the number of cells in the core $\mathcal{L}(H)$. As $\mathcal{P}(H)$ contains $O(n)$ vertices and $O(n^2)$ directed edges, one can then determine if there is a cycle in $\mathcal{P}(H)$ in $O(n^2)$ time.

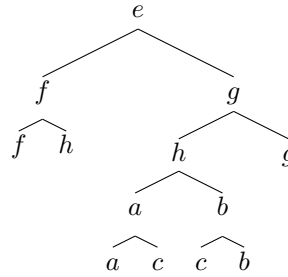
We remark that Bleak, Brough and Hermiller [7] have an algorithm for deciding the solvability of finitely generated computable subgroups of $\text{PL}_o(I)$ and in particular, of any

finitely generated subgroup of Thompson group F . The algorithm given by Corollary 11.25 applies the special properties of Thompson group F and cannot be generalized to subgroups of $\text{PL}_o(I)$. However, for finitely generated subgroups of F , it is much easier to implement than the algorithm in [7]. Indeed, the algorithm in [7] involves consideration of orbitals of elements in H and is composed of 4 steps, separated altogether into 29 sub-steps, most of which have to be iterated.

We finish this section with a demonstration of our algorithm for determining if a finitely generated subgroup of F is solvable.

Example 11.26. The subgroup $H = \langle x_0, x_1^2 x_2^{-1} x_1^{-1} \rangle$ of Thompson group F is solvable of derived length 2.

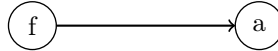
Proof. The core $\mathcal{L}(H)$ is given by the following minimal associated tree.



By considering trails with initial edges e, f, g, h, a, b and c which do not visit any edge of $\mathcal{L}(H)$ more than twice, we get that the periodic edges of $\mathcal{L}(H)$ are f with optimal trail $s_f \equiv 0$ and a with optimal trail $s_a \equiv 0$.

To find the outgoing edges of f in $\mathcal{P}(H)$, we consider all trails of the form $q \equiv u1w$ with initial edge f , such that $u0$ is a prefix of s_f and such that the sub-trail of q given by the suffix w is a simple trail. Since $s_f \equiv 0$, we must have $u \equiv \emptyset$. Thus, one has to consider trails $q \equiv 1w$ where the suffix w gives a simple sub-trail. One can check that starting from f , one can get via a trail $q \equiv 1w$ of this form to the vertices h, a, b and c . Hence, there is a directed edge in $\mathcal{P}(H)$ from f to a .

Since $s_a \equiv 0$, to find the outgoing edges of a in $\mathcal{P}(H)$, we again have to consider trails of the form $q' \equiv 1w'$ with initial edge a , where the suffix w' gives a simple sub-trail of q' . The only such trail (where $w' \equiv \emptyset$) has terminal edge c . Hence a has no outgoing edges in $\mathcal{P}(H)$. Thus, the graph $\mathcal{P}(H)$ is the following.



It follows from Theorem 11.24 that H is solvable of derived length 2. □

Remark 11.27. The group H from Example 11.26 is a copy of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ considered in [12]. The group H is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ because the support of $x_1^2 x_2^{-1} x_1^{-1}$ is contained in a fundamental domain of x_0 .

12 Open problems

12.1 Subgroups of F whose closure contains the derived subgroup of F

Let $H \leq F$. By Theorem 7.10, if (1) $[F, F] \leq \text{Cl}(H)$ and (2) there exists an element $h \in H$ which fixes a finite dyadic fraction $\alpha \in (0, 1)$ such that $h'(\alpha^+) = 2$ and $h'(\alpha^-) = 1$, then $H = F$. However, we do not have examples of subgroups H of F which satisfy the first condition but not the second. We note that the subgroups constructed in Theorem 9.1, Examples 10.12, and 10.13 and Lemma 10.15 were all constructed so as to satisfy condition (1) of Theorem 7.10 and simply “happened” to satisfy condition (2) as well.

Problem 12.1. Let $H \leq F$. Is it true that H contains the derived subgroup $[F, F]$ if and only if $\text{Cl}(H)$ contains $[F, F]$?

We note that if the answer to problem 12.1 is positive, then in particular, we have a positive answer to the following problem.

Problem 12.2. [15, Problem 5.12] Let H be a subgroup of F which is not contained in any proper finite index subgroup of F . Is it true that $H = F$ if and only if $\text{Cl}(H) = F$?

We note that if the answer to Problem 12.2 is positive, then it gives a simple algorithm for solving the generation problem in F .

12.2 Maximal subgroups of F of infinite index

12.2.A Closed maximal subgroups

All known maximal subgroups of F of infinite index are closed. Indeed, the Stabilizers $H_{\{\alpha\}}$ in F for $\alpha \in (0, 1)$ are closed for components, and as such, by Corollary 5.7, are closed subgroups. The maximal subgroup of F of infinite index constructed in [14] is also closed. In addition, the method for constructing maximal subgroups of F of infinite index demonstrated in Section 10.3 can only yield closed maximal subgroups.

Problem 12.3. [15, Problem 5.11] Is it true that every maximal subgroup of F of infinite index is closed?

Note that Problem 12.3 is equivalent to Problem 12.2. Indeed, it was observed in [15] that a positive answer to Problem 12.2 implies a positive answer to Problem 12.3. In the other direction, if F has a proper subgroup H which is not contained in any finite index subgroup of F and such that $\text{Cl}(H) = F$, then a maximal subgroup M of F containing H would be a counter example to Problem 12.3.

12.2.B The action of maximal subgroups of infinite index in F on $[0, 1]$

In Lemma 10.15 we constructed a maximal subgroup of infinite index in F which acts transitively on the set of finite dyadic fractions \mathcal{D} . The following problem remains open.

Problem 12.4. Does F have a maximal subgroup H of infinite index such that the orbits of the action of H on the interval $[0, 1]$ coincide with the orbits of the action of F ?

We note that if the answer to Problem 12.3 is negative, then the answer to Problem 12.4 is positive. Indeed, if H is a non-closed maximal subgroup of infinite index in F , then $\text{Cl}(H) = F$. Then by Corollary 5.8, the actions of H and of F on the interval $[0, 1]$ have the same orbits.

If $\alpha \in (0, 1)$ is a rational non-dyadic number, then $\alpha = .ps^{\mathbb{N}}$ where s is a minimal period. Clearly, s contains both digits 0 and 1. The orbit of α under the action of F is the set of all rational numbers in $(0, 1)$ with minimal period s . To decide if the action of H on this orbit is transitive, one can adapt the algorithm from Theorem 6.5. Indeed, we define a graph $\Gamma_s(H)$ whose vertex set is the vertex set of $\Gamma(H)$ (see Definition 6.3). There is a directed edge from e_1 to e_2 in $\Gamma_s(H)$ if and only if there is a trail labeled s with initial edge e_1 and terminal edge e_2 in $\mathcal{L}(H)$. Then Theorem 6.5, with the graph $\Gamma(H)$ replaced by $\Gamma_s(H)$ and the set \mathcal{D} replaced by the orbit of α under the action of F on $[0, 1]$, holds.

A consideration of the graph $\Gamma_s(H)$, where $s \equiv 01$ and H is the maximal subgroup of F from Lemma 10.15, shows that H does not act transitively on the orbit of $\frac{1}{3} = .(01)^{\mathbb{N}}$ under the action of F . Thus, H acts transitively on the set \mathcal{D} but is not a solution for Problem 12.4.

12.2.C 2-generated maximal subgroups of F

In [16], the author and Sapir prove that the isomorphism class of the stabilizer $H_{\{\alpha\}}$ of $\alpha \in (0, 1)$ depends only on the *type* of α ; i.e., on whether α is dyadic, rational non-dyadic or irrational. It is easy to see that if α is dyadic, then $H_{\{\alpha\}}$ is isomorphic to the direct product of two copies of F . Therefore, in that case, the minimal size of a generating set of $H_{\{\alpha\}}$ is 4. Savchuk observed that if α is irrational then $H_{\{\alpha\}}$ is not finitely generated. In [16] we prove that if α is finite dyadic then the stabilizer $H_{\{\alpha\}}$ has a generating set with 3 elements and does not have any smaller generating set. The explicit maximal subgroup of F constructed in [15] is isomorphic to F_3 ; the “brother group” of F which consists of all piecewise linear homeomorphisms of the unit interval $[0, 1]$ where all breakpoints are triadic fractions and all slopes are integer powers of 3. Thus, it has a generating set of size 3 and no less [9].

We note that all maximal subgroups of F constructed in this paper are 3- or 4- generated. Since the generating sets provided are not necessarily optimal, it is possible that one of them is 2-generated, but we do not know if that is the case. Note that the algorithm from [17, Lemma 9.11] enables to compute a presentation for the maximal subgroups constructed (as they are all closed). By moving to the abelianization, one can find a lower bound for the minimal size of a generating set. An application of this algorithm for the maximal subgroup from Example 10.12, shows that it cannot be generated by less than 3 elements. We did not apply the algorithm for the other maximal subgroups constructed in the paper. In general, the following problem is open.

Problem 12.5. Does Thompson group F have a 2-generated maximal subgroup of infinite index?

We note that by [3], for each prime p , Thompson group F has subgroups of index p which are isomorphic to F . Therefore, F has 2-generated maximal subgroups of finite index. In fact, we go as far as to make the following conjecture.

Conjecture 12.6. All finite index subgroups of F are 2-generated.

Note that all finite index subgroups of \mathbb{Z}^2 are 2-generated. Hence, the image of any finite index subgroup of F in the abelianization of F is 2-generated. The proof in Remark 9.3 of the group K being 2-generated, can be adapted to certain finite index subgroups of F considered by the author. We believe that the proof can be adapted to all finite index subgroups of F (note that Bleak and Wassink [3] found all isomorphism classes of finite index subgroups of F). It is interesting to note that if Conjecture 12.6 holds and the answer to Problem 12.5 is “no” then the minimal number of generators of a maximal subgroup of F is an invariant determining whether the subgroup has finite or infinite index in F .

12.3 Subgroups $H \leq F$ with finite core $\mathcal{L}(H)$

Problem 12.7. Let H be a subgroup of F such that the core $\mathcal{L}(H)$ is finite. Is it true that $\text{Cl}(H)$ is finitely generated?

We note that by Corollary 10.8, if $\mathcal{L}(H)$ is finite, then $\text{Cl}(H) = \text{Cl}(K)$, where K is finitely generated. Thus, Problem 12.7 is equivalent to [15, Problem 5.7], asking whether the closure of a finitely generated subgroup of F is finitely generated.

We note that by Theorem 11.1, the answer to Problem 12.7 is “yes” if H is solvable. If H is elementary amenable, we already do not have an answer to Problem 12.7.

If the core $\mathcal{L}(H)$ is finite then the semigroup presentation \mathcal{P} associated with $\mathcal{L}(H)$ is finite. It follows from Corollary 10.3, that if \mathcal{P} has a finite completion \mathcal{P}' (satisfying the conditions in Section 10.1), then $\text{Cl}(H)$ is finitely generated.

Problem 12.8. Let H be a subgroup of F with finite core $\mathcal{L}(H)$. Let \mathcal{P} be the semigroup presentation associated with $\mathcal{L}(H)$. Is it true that \mathcal{P} has a finite completion \mathcal{P}' (satisfying the conditions from Section 10.1)?

We find it unlikely that the answer to Problem 12.8 would be “yes”. But as Propositions 10.1 and 10.2 show, semigroup presentations \mathcal{P} associated with cores $\mathcal{L}(H)$ of subgroups $H \leq F$ satisfy some special properties.

Finally, we note that there are finitely generated closed subgroups of F which are not finitely presented. For example, the subgroup from Example 11.26, isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, is a closed subgroup of F . Indeed, it is easy to prove that it is closed for components and therefore closed by Corollary 5.7. The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is 2-generated but is not finitely presented.

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